\textbf{\$-MANIFOLDS AND CONE-DUAL MAPS}

SARA DRAGOTTI AND GAETANO MAGRO

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\textbf{Abstract.} Let $f: P \rightarrow Q$ be a simplicial map such that $D(\alpha, f)$, the dual to $\alpha$ with respect to $f$, is a cone, for each simplex $\alpha$ of $Q$. It is shown that if $P$ is an \$-manifold then $f$ is approximable by PL-homeomorphisms, provided that $f$ satisfies an extra condition on the boundary of $P$.

\textbf{Introduction.} An interesting area of research has been that of trying to identify those maps which are approximable by PL-homeomorphisms or topological homeomorphisms. The domain and the range of these maps are spaces with extra structures, as PL-manifolds, homology manifolds, polyhedra etc. So, for example, a cellular map $f: M \rightarrow N$ between topological $n$-manifolds is approximable by homeomorphisms (see Siebenmann [11], for $n \neq 4, 5$. The referee pointed out to us that Freedman and Edwards have proved the result for $n = 4$ and $n = 5$, respectively).

A generalization of this result to homology manifolds by introducing a more general concept of cellularity can be found in Henderson, [7].

If $M$ is a PL-manifold, a cellular map, or PL-cellular map, it is not approximable by PL-homeomorphisms. A class of maps (transversely cellular maps) which do this is given by M. Cohen in [3].

In a recent work, [5], we have studied the problem when the domain is a homology manifold, and we have found a class of maps, called cone-dual maps, preserving homology manifold's structure, but nonpreserving the PL-homeomorphism's class. In the attempt to increase the last definition in order to obtain the required approximability, we arrive at defining the strong cone-dual maps. These last maps are approximable by PL-homeomorphisms when even the domain is a simple polyhedron.

In the present work we investigate the approximability by a PL-homeomorphism or top-homeomorphism, of maps between \$-manifolds. The \$-manifolds are a class of polyhedra which includes homology manifolds without boundary or with collared boundary.

The results obtained may be summarized as follows:

(1) A cone-dual map $f: M \rightarrow N$ is approximable by a PL-homeomorphism where $M$ is an \$-manifold without boundary (Theorem 3.1).

(2) If $M$ is an \$-manifold with boundary $\partial M$, a cone-dual map $f: M \rightarrow N$ is approximable by a PL-homeomorphism provided $f(\partial M)$ is collared in $N$ (Theorem 3.2).

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The condition "f(∂M) is collared in N" is necessary to approximate f to a top-homeomorphism, i.e. there exist cone-dual maps f: M → N where M is an \( \mathcal{L} \)-manifold with boundary and N is not topological homeomorphic to M (§4).

1. Cone-dual maps. Let \( K \) be a simplicial complex, for each simplex \( \alpha \) of \( K \), the dual to \( \alpha \) in \( K \), denoted \( D(\alpha, K) \), and its subcomplex \( \hat{D}(\alpha, K) \) are defined by

\[
D(\alpha, K) = \{ b(\sigma_1) \cdots b(\sigma_h) | \alpha < \sigma_1 < \cdots < \sigma_h < K \},
\]

and

\[
\hat{D}(\alpha, K) = \{ b(\sigma_1) \cdots b(\sigma_h) | \alpha \leq \sigma_1 < \cdots < \sigma_h < K \}
\]

where \( b(\sigma_i) \) denotes the barycenter of \( \sigma_i \).

It is known that

(a) \( D(\alpha, K) = b(\alpha) \ast \hat{D}(\alpha, K) \)

(b) \( \hat{D}(\alpha, K) \cong_{PL} Lk(\alpha, K) \)

Throughout this paper all polyhedra are assumed to be compact and connected.

Let \( f: K \to L \) be a simplicial map, \( L' \) the first barycentric subdivision of \( L \), and \( K' \) a barycentric subdivision of \( K \) chosen so that \( f \) is also simplicial with respect to \( K' \) and \( L' \). For each simplex \( \alpha \) of \( L \), \( D(\alpha, f) \) and \( \hat{D}(\alpha, f) \) are the subcomplexes of \( K' \) defined by

\[
D(\alpha, f) = \{ b(\sigma_1) \cdots b(\sigma_h) | \alpha < f(\sigma_1), \sigma_1 < \cdots < \sigma_h < K \},
\]

and

\[
\hat{D}(\alpha, f) = \{ b(\sigma_1) \cdots b(\sigma_h) | \alpha \leq f(\sigma_1), \sigma_1 < \cdots < \sigma_h < K \}
\]

\( D(\alpha, f) \) is called the dual to \( \alpha \) with respect to \( f \).

We refer to [2, 3] for the proofs of following results.

**Proposition 1.1.** \( D(\alpha, f) = f^{-1}(D(\alpha, L)) \); \( \hat{D}(\alpha, f) = f^{-1}(\hat{D}(\alpha, L)) \).

**Proposition 1.2.** \( K \) is the union of the duals of the simplexes of \( L \) with respect to \( f \). Moreover we have:

(a) \( \hat{D}(\alpha, f) = \bigcup_{\alpha < \beta} D(\beta, f) \),

(b) \( D(\alpha, f) \cap D(\beta, f) = D(\alpha \cdot \beta, f) \),

where \( \alpha \cdot \beta \) is the simplex spanned by \( \alpha \) and \( \beta \) if there is one, \( \alpha \cdot \beta = \emptyset \) otherwise, and \( D(\emptyset, f) = \emptyset \).

**Proposition 1.3.** If \( \alpha^{i-1} < \alpha^i \), then \( D(\alpha^i, f) \) is a regular neighbourhood of \( f^{-1}(b(\alpha^i)) \) in \( \hat{D}(\alpha^{i-1}, f) \) with boundary \( D(\alpha^i, f) \). (For \( i = 0 \) we assume \( D(\alpha^{-1}, f) = K' \).)

We refer to [5] for the following definition and proposition.

A simplicial map \( f: K \to L \) is called strong cone-dual if \( (D(\alpha, f), \hat{D}(\alpha, f)) \) is a cone pair for each simplex \( \alpha \) of \( L \), i.e. there is a PL-homeomorphism of \( D(\alpha, f) \) onto a cone on \( \hat{D}(\alpha, f) \), which maps \( \hat{D}(\alpha, f) \) on \( D(\alpha, f) \) (see Stallings [12]).

**Proposition 1.4.** If \( f: K \to L \) is a surjective strong cone-dual map, then there is a PL-homeomorphism \( \hat{f} \) of \( K \) into \( L \) such that \( \hat{f}(D(\alpha, f)) = D(\alpha, L) \), for each simplex \( \alpha \) of \( L \).

In [5] we have defined cone-dual maps between homology manifolds. It is possible to extend this definition to maps between polyhedra as soon as we define the boundary of a polyhedron.
Given an $h$-polyhedron $P$, the boundary of $P$ is the subpolyhedron defined inductively by

$$
\partial P = \begin{cases} 
\emptyset, & h = 0, \\
\{ x \in P | \text{Lk}(x, P) = \text{point or } \partial \text{Lk}(x, P) \neq \emptyset \}, & h \geq 1,
\end{cases}
$$

**Definition 1.5.** A simplicial map $f: P \rightarrow Q$ is called cone-dual with respect to the triangulations $K$ and $L$ of the polyhedra $P$ and $Q$ if $D(\sigma, f)$ is a cone for each simplex $\sigma$ of $f(K)$, and $D(\sigma, f/\partial K)$ is a cone for each simplex $\sigma$ of $f(\partial K)$.

The next theorem shows that Definition 1.5 does not depend on the triangulations chosen.

**Theorem 1.6.** Let $f: K \rightarrow L$ be a cone-dual map, $K$ and $L$ triangulations of $K$ and $L$ such that $f$ is also simplicial. Then $f$ is cone-dual with respect to $K$ and $L$.

To prove this theorem we will use the following lemma.

**Lemma 1.7.** Let $f: K \rightarrow L$ be a cone-dual map, if $\beta^i$ is an $i$-simplex of $f(K)$ and $\beta^j$ is a $j$-face of $\beta^i$, then a regular nbd of $f^{-1}(b(\beta^i))$ in $D(\beta^j, f)$ is PL-homeomorphic to a cone. (If $\beta^j = \emptyset$ assume $D(\beta^j, f) = K'$.)

**Proof.** If $j = i - 1$, the result follows from Proposition 1.3. So we suppose $j < i - 1$.

Let $\beta^{i+1}, \beta^{i+2}, \ldots, \beta^{i-1}$ be a finite sequence of simplexes of $f(K)$ such that $\beta^i < \beta^{i+1} < \cdots < \beta^{i-1} < \beta^j$. By Proposition 1.3, $D(\beta^h, f)$ is a regular nbd of $f^{-1}(b(\beta^h))$ in $D(\beta^{i-1}, f)$ with boundary $D(\beta^h, f)$. Hence $D(\beta^h, f)$ is bicollared in $D(\beta^{i-1}, f)$. This implies that a regular nbd $U$ of $D(\beta^h, f)$ in $D(\beta^j, f)$ is PL-homeomorphic to $D(\beta^{i-1}, f) \times [-1, 1]^r$ ($r = i - j - 1$), by a PL-homeomorphism $\varphi: D(\beta^{i-1}, f) \times [-1, 1]^r \rightarrow U$ so that $\varphi(\hat{D}(\beta^{i-1}, f) \times \{ 0 \}^r) = \hat{D}(\beta^j, f)$. Since $D(\beta^j, f)$ is a regular nbd of $f^{-1}(b(\beta^j))$ in $D(\beta^{i-1}, f)$, a regular nbd of $f^{-1}(b(\beta^j))$ in $D(\beta^j, f)$ is PL-homeomorphic to $D(\beta^j, f) \times [-1, 1]^r$, which is a cone. □

**Proof of Theorem 1.6.** First suppose that $L$ is obtained from $L$ by starring at only point $v = b(\bar{v})$. Generally for each simplex $\alpha$ of $L$, by $\bar{\alpha}$ we mean the carrier of $\alpha$ in $L$. Moreover we denote by $\bar{D}(\alpha, f)$ and by $\bar{D}(\alpha, f)$ the corresponding carrier of $D(\alpha, f)$ and $\hat{D}(\alpha, f)$ with respect to $f: K \rightarrow L$, i.e.: $\bar{D}(\alpha, f) = f^{-1}(D(\alpha, \bar{L}))$, $\hat{D}(\alpha, f)) = f^{-1}(D(\alpha, \bar{L}))$.

To prove that $\bar{D}(\alpha, f)$ is a cone for each $\alpha$ of $L$, we proceed to consider the various cases.

**Case 1.** Assume $\alpha \in \bar{L} \cap L$.

Note that if $\alpha$ does not lie in $\text{Lk}(v, \bar{L})$, we have $D(\alpha, \bar{L}) = D(\alpha, L)$, hence $\bar{D}(\alpha, f) = D(\alpha, f)$.

If $\alpha$ is a face of some simplex which contains $v$, then there exists a PL-homeomorphism between $\bar{D}(\alpha, f)$ and $D(\alpha, f)$. In fact let $\varphi(b(\tau)) = b(\bar{\tau})$ for each vertex $b(\tau)$

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1 An equivalent definition of boundary can be found in [13].
of \( \overline{D}(\alpha, f) \). Evidently, if \( \alpha \), in \( \overline{L} \), is a face of \( f(\tau) \), then \( \alpha \), as simplex of \( L \), will be a face of \( f(\tilde{\tau}) \). Hence \( b(\tilde{\tau}) \) is a vertex of \( D(\alpha, f) \). It follows that \( \varphi \) carries the vertices of \( \overline{D}(\alpha, f) \) into vertices of \( D(\alpha, f) \).

\( \varphi \) is an injective map. In fact if \( \varphi(b(\tau)) = \varphi(b(\delta)) \), then \( \tau \) and \( \delta \) are simplexes of \( \overline{K} \) so that \( \tilde{\tau} = \tilde{\delta} \). This implies either \( \tau = \delta \), or \( v \) lies in \( \alpha (\alpha < f(\tau), \alpha < f(\delta)) \). Since \( \alpha \in \overline{L} \cap L \), the last eventuality does not occur.

Trivially \( \varphi \) is surjective, and hence bijective.

Observe that if \( \tau_1 < \tau_2 \), then \( \tilde{\tau}_1 < \tilde{\tau}_2 \). Hence if \( b(\tau_1), \ldots, b(\tau_h) \) lie in a simplex of \( \overline{D}(\alpha, f) \), then \( b(\tilde{\tau}_1), \ldots, b(\tilde{\tau}_h) \) lie in a simplex of \( D(\alpha, f) \). Therefore \( \varphi \) can be extended to a PL-homeomorphism, which we will denote again by \( \varphi \), of \( \overline{D}(\alpha, f) \) onto \( D(\alpha, f) \).

Observe that, in this case, if \( \alpha \preceq f(\tau) \) then \( \alpha \preceq f(\tilde{\tau}) \). Consequently \( \varphi \) takes \( \overline{D}(\alpha, f) \) onto \( D(\alpha, f) \), and \( \varphi \) coincides with identity when \( \alpha \) does not lie in the closure of the star of \( v \) in \( \overline{L} \).

Thus if \( \alpha \in L \cap \overline{L} \), then \( \overline{D}(\alpha, f) \), being PL-homeomorphic to the cone \( D(\alpha, f) \), is a cone.

Case II. Assume \( \alpha \in \overline{L} - L \), \( \alpha \neq v \).

In this case \( v \) is a vertex of \( \alpha \). Since \( \alpha \neq v \), there exists a simplex \( \beta \in L \cap \overline{L} \) so that \( \beta \) is a 1-codimensional face of \( \alpha \) (\( \beta = \) opposite face to \( v \)). Then \( \overline{D}(\alpha, f) \) is a regular nbd of \( f^{-1}(b(\alpha)) \) in \( \overline{D}(\beta, f) \). From the previous case it follows that \( \overline{D}(\beta, f) \) is PL-homeomorphic to \( \overline{D}(\alpha, f) \). On the other hand \( f^{-1}(b(\alpha)) \) is PL-homeomorphic to \( f^{-1}(b(\tilde{\alpha})) \). Now observe that \( \beta \) is also a face of \( \tilde{\alpha} \), but in general it is not a 1-codimensional face. However, by Lemma 1.7, we can assert that \( f^{-1}(b(\tilde{\alpha})) \) has in \( \overline{D}(\beta, f) \) a regular nbd which is a cone. Thus, from the uniqueness theorem for regular nbd, it follows that \( \overline{D}(\alpha, f) \) is a cone.

Case III. Assume \( \alpha = v \).

By Lemma 1.7, \( f^{-1}(v) = f^{-1}(b(\tilde{v})) \) has a regular nbd which is a cone in \( K' \). As above, using the uniqueness theorem for regular nbd, and the fact that \( K' \) is also a subdivision of \( \overline{K} \), we have that \( \overline{D}(v, f) \) is a cone.

In order to prove the result in the general case, we must show that, assuming \( L \) and \( \overline{L} \) as above, we can exchange their roles with respect to \( f \). That is, if we suppose that \( \overline{D}(\alpha, f) \) is a cone for every simplex \( \alpha \) of \( \overline{L} \), then \( D(\alpha, f) \) is a cone for every \( \alpha \in L \).

In fact, if \( \alpha \in \overline{L} \cap L \), we have proved that \( D(\alpha, f) \) is PL-homeomorphic to \( \overline{D}(\alpha, f) \), and hence \( D(\alpha, f) \) is a cone. If instead \( \alpha \) lies in \( L - \overline{L} \), then it is the carrier of a simplex \( \overline{\alpha} \) of \( \overline{L} \) of the same dimension. Then, if \( \beta \) is a simplex of \( L \cap \overline{L} \) and a 1-codimensional face of \( \alpha \) and \( \overline{\alpha} \), reasoning as before (Case II), we have that \( f^{-1}(b(\alpha)) \) has a regular nbd in \( \overline{D}(\beta, f) \) which is a cone. Hence \( D(\alpha, f) \) is a cone.

Finally, to complete the proof, it suffices to recall that equivalent triangulations of \( L \) have a common subdivision, and that every subdivision \( L' \) of \( L \) can be obtained from \( L \) by a finite number of subdivisions \( L' = L_h < L_{h-1} < \cdots < L_1 = L \) so that \( L_i \) is obtained from \( L_{i+1} \) by introducing an only vertex. \( \square \)

2. Duals and \( \mathcal{L} \)-manifolds. In this section we investigate the dual structure induced by a simplicial map \( f : K \to L \), on \( K \), when \( K \) is an \( \mathcal{L} \)-manifold.

For the reader's convenience, we reproduce here the definition of \( \mathcal{L} \)-manifold, according to [1].
Suppose we are given a class $\mathcal{L}_n$, for each $n \geq 0$, of $(n - 1)$-polyhedra (closed under PL-homeomorphisms), which satisfies:

1. Each member of $\mathcal{L}_n$ is a polyhedron whose links lie in $\mathcal{L}_{n-1}$.
2. $\Sigma \mathcal{L}_{n-1} \subseteq \mathcal{L}_n$ (i.e. the suspension of an $(n - 1)$-link is an $n$-link).
3. $c\mathcal{L}_{n-1} \cap \mathcal{L}_n = \emptyset$ (i.e. the cone on $(n - 1)$-link is never a link).

Then an $\mathcal{L}_n$-manifold $M$ is a polyhedron whose links lie either in $\mathcal{L}_n$ or $c\mathcal{L}_{n-1}$. The boundary of $M$, $\partial M$, consists of points whose links lie in the latter class.

As an immediate consequence of the definition we observe that the boundary of an $\mathcal{L}_n$-manifold $M$ is itself an $\mathcal{L}_{n-1}$-manifold $\partial M$ without boundary. Furthermore $\partial M$ is collared in $M$.

**Remark 2.1.** The link of an $i$-simplex in an $\mathcal{L}_n$-manifold lies either in $\mathcal{L}_{n-i}$ or in $c\mathcal{L}_{n-i}$.

**Remark 2.2.** Every polyhedron of $\mathcal{L}_n$ is an $\mathcal{L}_{n-1}$-manifold without boundary.

**Remark 2.3.** An $\mathcal{L}$-manifold, which is a cone, is a cone on the complete boundary.

In fact, let $M = v \ast H$ be an $\mathcal{L}$-manifold. Since $H = \text{Lk}(v, M)$, $H$ lies either in $\mathcal{L}_n$ or in $c\mathcal{L}_{n-1}$. If $H$ lies in $\mathcal{L}_n$, then $\partial H = \emptyset$. Hence $H$ is the complete boundary of $M$. If instead $H = cA$, with $A \in \mathcal{L}_{n-1}$, then we have: $M = ccX \equiv_{PL} c\Sigma X$ and $\partial(c\Sigma X) = \Sigma X \equiv_{PL} \partial M$. This implies $M \equiv_{PL} c\partial M$.

**Theorem 2.4.** Let $f: M \to L$ be a simplicial map, where $M$ is an $\mathcal{L}_n$-manifold and $L$ a polyhedron. For each $i$-simplex $\gamma$ of $f(M)$, $D(\gamma, f)$ is an $\mathcal{L}_{n-1}$-manifold with boundary $\hat{D}(\gamma, f) \cup D(\gamma, f/\partial M)$.

**Proof.** Let $\sigma = b(\sigma_0) \cdots b(\sigma_h)$ be a simplex of $D(\gamma, f) - \hat{D}(\gamma, f)$, we have that (see [3])

$$\text{Lk}(\sigma, D(\gamma, f)) = \{b(\tau_0) \cdots b(\tau_q) | \gamma = f(\tau_0) = \cdots = f(\tau_q); \tau_q < \sigma_0 \}$$
$$\ast \hat{D}(\sigma_0, \sigma_1) \ast \cdots \ast \hat{D}(\sigma_{h-1}, \sigma_h) \ast \hat{D}(\sigma_h, M)$$
$$= [(f/\sigma_0)^{-1}(b(\gamma))] \ast \hat{D}(\sigma_0, \sigma_1) \ast \cdots \ast \hat{D}(\sigma_{h-1}, \sigma_h) \ast \hat{D}(\sigma_h, M).$$

Let $\dim \sigma_h = n_h$, using results about duals in a PL-manifold, it follows that $\text{Lk}(\sigma, D(\gamma, f)) = \Sigma^r \text{Lk}(\sigma_h, M)$, where $r = n_h - h - i$. Therefore $\text{Lk}(\sigma, D(\gamma, f))$ lies either in $\mathcal{L}_{n-i-h}$ or $c\mathcal{L}_{n-i-h-1}$, depending on whether $\sigma_h$ (and hence $\sigma$) is in $M - \partial M$ or not.

Suppose now that $\sigma$ lies in $\hat{D}(\gamma, f)$, we have

$$\text{Lk}(\sigma, D(\gamma, f)) = D(\gamma, f/\sigma_0) \ast \hat{D}(\sigma_0, \sigma_1) \ast \cdots \ast \hat{D}(\sigma_{h-1}, \sigma_h) \ast \hat{D}(\sigma_h, M).$$

Since $D(\gamma, f/\sigma_0)$ is a PL-ball (see [3]), it follows that

$$\text{Lk}(\sigma, D(\gamma, f)) \equiv_{PL} v \ast \Sigma^{k+1} \text{Lk}(\sigma_h, M), \quad \text{where } k = n_h - i - h - 2.$$

If $\sigma_h$ does not lie in $\partial M$, then $\text{Lk}(\sigma_h, M)$ lies in $\mathcal{L}_{n-n_h-1}$ and hence $\text{Lk}(\sigma_h, D(\gamma, f))$ lies in $c\mathcal{L}_{n-i-h-1}$. If $\sigma_h$ lies in $\partial M$, then $\text{Lk}(\sigma_h, M)$ is a cone on a polyhedron $X$ of $\mathcal{L}_{n-n_h-1}$, it follows that

$$\text{Lk}(\sigma, D(\gamma, f)) \equiv_{PL} v \ast c\Sigma^{k+1} X \equiv_{PL} c\Sigma^{k+2} X.$$

So we have that $D(\gamma, f)$ is an $\mathcal{L}_{n-i}$-manifold.
Moreover Lk(σ, D(γ, f)) is a cone if and only if σ lies either in ∂M or in ̃D(γ, f).
This implies that
\[ \partial D(γ, f) = ̃D(γ, f) \cup D(γ, f/∂M). \]

3. Cone-dual maps and \( L \)-manifolds. Let \( f: M \to N \) be a cone-dual map. In \([5]\) we have proved that if \( M \) is a homology manifold, then \( f(M) \) is itself a homology manifold. But in general \( f(M) \) is not PL-homeomorphic to \( M \).

The next theorem shows that this result can be improved if we suppose that \( M \) is a homology manifold, or, more generally an \( L \)-manifold, without boundary.

**THEOREM 3.1.** Let \( f: M \to N \) be a surjective cone-dual map. If \( M \) is an \( L \)-manifold without boundary, then \( F \) is approximable by a PL-homeomorphism.

**PROOF.** By Proposition 1.4 and Theorem 1.6 it suffices to prove that \( f \) is a strong cone-dual map.

Let \( η \) be a simplex of \( N \). By hypothesis \( D(η, f) \) is PL-homeomorphic to a cone \( cX \). On the other hand, from Theorem 2.4, \( D(η, f) \) is an \( L \)-manifold with boundary \( ̃D(η, f) \). Hence, using Remark 2.3, we can suppose \( X = ̃D(η, f) \). This implies that \((D(η, f), ̃D(η, f))\) is a cone pair. □

Note that the condition “\( ∂M = \emptyset \)” cannot be dispensed in order to obtain the last result. In fact, let \( Q^n \) be a contractible PL \( n \)-manifold whose boundary \( ∂Q^n \) is a homology \((n - 1)\)-sphere\(^2\) not simply connected. Such examples are known to exist for \( n \geq 5 \) (see \([4]\)). Now let \( M^n \) be the homology \( n \)-sphere defined by
\[ M^n = (v \circ ∂Q^n) \cup Q^n \]
and let \( K \) and \( L \) be the homology \((n + 1)\)-manifolds
\[ K = c \circ M^n, \quad L = v \circ Q^n. \]

One can see that the simplicial map defined on the set of vertices of \( K \) by setting:
\[ f(v_i) = v_i, \text{ if } v_i \neq c, \quad f(c) = v \]
is cone-dual (see \([5]\)). On the other hand, \( K \) is a homology manifold with collared boundary, hence \( K \) is an \( L \)-manifold with boundary \((L_n = \text{homology } (n - 1)\)-spheres, \( n \geq 0 \)), while \( L \) is not an \( L \)-manifold.

Note that \( f(∂K) = ∂L \) is not collared in \( L \).

From the above arguments, it seems natural to ask if the Theorem 3.1 can be extended to the case of an \( L \)-manifold with boundary nonempty as soon as \( f: M \to N \) satisfies the extra condition “\( f(∂M) \) is collared in \( N \)”.

The next theorem gives an affirmative answer to this question. It will follow that there are not cone-dual maps which preserve \( L \)-manifold’s structure, but do not preserve the PL-homeomorphism’s class.

**THEOREM 3.2.** Let \( f: M \to N \) be a surjective cone-dual map, where \( M \) is an \( L \)-manifold with boundary \( ∂M \). If \( f(∂M) \) is collared in \( N \), then \( f \) is approximable by a PL-homeomorphism.

**PROOF.** Let \( h = \dim M \). From the hypothesis and the previous theorem it follows that \( N' = f(∂M) \) is an \( L_{h-1} \)-manifold collared in \( N \). Consequently \( N \) also has dimension \( h \).

\(^2\)By a homology \( n \)-sphere we mean a homology \( n \)-manifold which has the same homology as an \( n \)-sphere.
Now we will prove that \((D(\tau, f), \hat{D}(\tau, f))\) is PL-homeomorphic to \((D(\tau, N), \hat{D}(\tau, N))\) for each simplex \(\tau\) of \(N\), proceeding by decreasing induction on dimension of the simplexes of \(N\).

Let \(\tau\) be an \(h\)-simplex of \(N\). Then \(D(\tau, N) = b(\tau)\) and \(\hat{D}(\tau, N) = \emptyset = \hat{D}(\tau, f)\). Because \(D(\tau, f/\partial M) = \emptyset\), \(D(\tau, f)\) is just one point. This implies that \(D(\tau, f)\) is PL-homeomorphic to \(D(\tau, N)\).

Suppose now that \(\tau\) is an \((h - 1)\)-simplex of \(N\). We will prove that there exists a PL-homeomorphism

\[
\varphi_\tau : (D(\tau, f), \hat{D}(\tau, f)) \rightarrow (D(\tau, N), \hat{D}(\tau, N))
\]

so that the following conditions hold:

(i) \(\sigma < \tau \Rightarrow \varphi_\tau / D(\sigma, f) = \varphi_\sigma\),

(ii) \(\tau \in N' \Rightarrow \varphi_\tau(D(\tau, f/\partial M)) = D(\tau, N')\).

There are two cases according to whether \(\tau \in N'\), or not.

Case I. Assume \(\tau \notin N' = f(\partial M)\).

From above and Proposition 1.2, we have

\[
D(\tau, f) = \bigcup_{\sigma < \tau} D(\sigma, f) \quad \Phi \bigcup_{\sigma < \tau} D(\sigma, N) = \hat{D}(\tau, N).
\]

Since in this case \(D(\tau, f)\) and \(D(\tau, N)\) are cones on \(\hat{D}(\tau, f)\) and \(\hat{D}(\tau, N)\) respectively, one can extend \(\varphi\) conewise to obtain the desired PL-homeomorphism.

Case II. Assume \(\tau \in N'\).

By hypothesis there exists one only \(h\)-simplex \(\sigma^h\) of \(N\) such that \(\tau < \sigma^h\). Then we have: \(\hat{D}(\tau, N) = D(\sigma^h, N) = b(\sigma^h)\). This implies that: \(\hat{D}(\tau, f) = f^{-1}(D(\sigma^h, N)) = D(\sigma^h, f)\). Hence \(\hat{D}(\tau, f)\) is a single point. On the other hand \(D(\tau, f/\partial M)\) is also a single point. It follows that \(D(\tau, f)\) is a cone on two points. Being \(D(\tau, N)\) a cone over a point, it is trivial to construct the required \(\varphi_\tau\).

Now we suppose that for all simplexes \(\tau\) of \(N\) of dimension greater than \(d\) there exists a PL-homeomorphism

\[
\varphi_\tau : (D(\tau, f), \hat{D}(\tau, f)) \rightarrow (D(\tau, N), \hat{D}(\tau, N))
\]

satisfying (i) and (ii). Let \(\tau\) be a \(d\)-simplex of \(N\).

As above we distinguish two cases. If \(\tau \notin N'\), it follows

\[
\partial D(\tau, f) = \hat{D}(\tau, f) = \bigcup_{\sigma < \tau} D(\sigma, f) \quad \Phi \bigcup_{\sigma < \tau} D(\sigma, N) = \hat{D}(\tau, N)
\]

where \(\Phi\) is the PL-homeomorphism obtained by gluing the PL-homeomorphisms \(\varphi_\sigma\) as stated by inductive hypothesis. The required PL-homeomorphism is obtained by conical extension of \(\Phi\).

If instead \(\tau \in N'\) we see that

\[
\hat{D}(\tau, f) = \bigcup_{\tau < \sigma} D(\sigma, f) \quad \Phi \bigcup_{\tau < \sigma} D(\sigma, N) = \hat{D}(\tau, N).
\]

Then (ii) implies that

\[
\Phi(\partial D(\tau, f/\partial M)) = \Phi \left( \bigcup D(\sigma, f/\partial M) \right) = \bigcup D(\sigma, N') = \hat{D}(\tau, N).
\]
By extending $\Phi/\hat{D}(\tau, f/\partial M)$ conewise, we obtain a PL-homeomorphism $\Phi': D(\tau, f/\partial M) \rightarrow D(\tau, N')$. Now we denote by $\overline{\Phi}$ the PL-homeomorphism obtained by gluing $\Phi$ and $\Phi'$. Because $N' = f(\partial M)$ is collared in $N$, we have that $\hat{D}(\tau, N) = c\hat{D}(\tau, N')$. Since $D(\tau, N')$ also is a cone on $\hat{D}(\tau, N')$, then we have

$$\hat{D}(\tau, N) \cup D(\tau, N') \cong D(\tau, N') = \Sigma\hat{D}(\tau, N').$$

Thus it follows

$$D(\tau, N) = c\hat{D}(\tau, N) \cong c\Sigma\hat{D}(\tau, N') \cong c * (\hat{D}(\tau, N) \cup D(\tau, N')).$$

The desired PL-homeomorphism between $D(\tau, f)$ and $D(\tau, N)$ can be obtained by extending $\overline{\Phi}$ conewise.

Thus we have proved that $(D(\tau, f), \hat{D}(\tau, f))$ is a cone pair for each simplex $\tau$ of $N$. Applying 1.4 and Theorem 1.6, the assertion follows. \(\Box\)

4. A counterexample. In the previous section we have seen that the property of being cone-dual for a map defined on an $\mathcal{L}$-manifold with boundary does not suffice by itself to preserve the PL-homeomorphism’s class. It is natural to ask if a cone-dual map preserves at least the topological homeomorphism’s class.

The following example shows this to be false.

Let $Q$ be the Mazur homology 3-sphere. Assume $v$ a vertex of $Q$, and denote by $P$ the PL-manifold obtained from $Q$ by removing the open star of $v$. Let $K = c * Q$ and $L = v * P$.

We see that $K$ and $L$ are not homeomorphic.

In fact, because the suspension of $Q$ is not homeomorphic to the 4-sphere, $K - \partial K$ is not a topological manifold, and $c$ is a singular point. On the other hand, clearly, $L - \partial L$ is a PL-manifold. Hence a possible homeomorphism of $K$ in $L$ will take the point $c$ in a point of $\partial L$. But this is excluded by local homology’s arguments.

Finally we can easily observe that the simplicial map $\varphi$ of $K$ in $L$ defined on the set of vertices of $K$ by setting: $\varphi(c) = v$ and $\varphi(v_1) = v_i$ if $v_i \neq c$, is cone-dual.

REFERENCES

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