ABSTRACT. Let $H(A)$ be the Dowker's generalized Hilbert space with weight $|A|$, where $A$ is any infinite set, and $H_{\infty}(A)$ its subspace consisting of all points which have only finitely many rational coordinates distinct from zero. Using a result of E. Pol, it will be shown that $H_{\infty}(A)$ is a universal space for countable dimensional metric spaces with weight $\leq |A|$.

1. Introduction. A metrizable space $X$ is called countable dimensional if it can be represented as the union of a sequence $X_1, X_2, \ldots$ of subspaces such that $\dim X_i \leq 0$, $i = 1, 2, \ldots$, where $\dim$ denotes the covering dimension. Universal spaces for countable dimensional metric spaces were studied by J. Nagata and it was shown that

(a) the subspace $N^\omega$ of the Hilbert cube $I^\omega$ consisting of all points which have at most finitely many rational coordinates is a universal space for countable dimensional separable metric spaces [3, Corollary 4.4], and

(b) the subspace $K_{\infty}(A)$ of the Cartesian product $P(A)$ of $\aleph_0$ copies of the star space $S(A)$ is a universal space for countable dimensional metric space with weight $\leq |A|$, where $|A|$ denotes the cardinality of $A$ and $K_{\infty}(A)$ consists of all points in $P(A)$ which have only finitely many rational coordinates distinct from zero [4, Theorem 2].

On the other hand, the generalized Hilbert space $H(A)$ was shown to be universal for all metrizable space with weight $\leq |A|$ by C. H. Dowker [1, Lemma 1]. The purpose of this note is to show the following result.

**Theorem.** Let $H_{\infty}(A)$ be the set consisting of all points in $H(A)$ at most finitely many of whose coordinates are rational different from zero. Then $H_{\infty}(A)$ is a universal space for countable dimensional metric spaces with weight $\leq |A|$.

2. Proof of Theorem. Let $A$ be any infinite set and $R$ the set of real numbers. Then the Hilbert space $H = H(A)$ can be defined as a space of all points $x$ in $R^A$ such that $\sum_\alpha x(\alpha)^2 < \infty$, and its norm is defined by $||x|| = (\sum_\alpha x(\alpha)^2)^{1/2}$. We denote by $B_\delta(x)$ the spherical neighborhood of $x$ with radius $\delta$, i.e. $B_\delta(x) = \{x' \in H : d(x, x') < \delta \}$ where $d$ is the metric defined by $d(x, x') = ||x - x'||$. For every integer $n \geq 0$, let $H_n = H_n(A)$ be the set of points in $H$ exactly $n$ of whose coordinates are nonzero rationals. Thus the space $H_0$ consists of all points in $H$ whose nonzero coordinates are irrational, and E. Pol established the equality

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\[ \dim H_0 = 1 \] [4, Example 1.6]. Using this result we now prove

**Lemma 1.** \( \dim H_n = 1 \) for every integer \( n \geq 0 \).

**Proof.** First we note that

1. if \( J \) is any interval in the real line \( R \) such that \( 0 \notin \bar{J} \) (the closure of \( J \)) and \( x \) is a point in \( H \), then the set \( \{ \alpha \in A : x(\alpha) \in J \} \) is finite.

Now we assume that \( A \) is well ordered with an ordering relation \(<\), and denote by \( A_n, n \geq 1 \), the family of all subsets of \( A \) which consist of exactly \( n \) elements. Thus every \( \xi \) in \( A_n \) can be uniquely expressed as \( \xi = (\alpha_1, \ldots, \alpha_n) \) with \( \alpha_1 < \alpha_2 < \cdots < \alpha_n \). Let \( Q_0 \) be the set of nonzero rationals and \( P_0 = R \setminus Q_0 \), i.e. the set of irrational and zero. For each \( \rho = (r_1, \ldots, r_n) \in Q_0^n \) and \( \xi = (\alpha_1, \ldots, \alpha_n) \in A_n \), we define

\[ H(\rho, \xi) = \{ x \in H_n : x(\alpha_i) = r_i, \ i = 1, \ldots, n \}. \]

Note that \( x(\alpha) \in P_0 \) for each \( x \in H(\rho, \xi) \) and \( \alpha \in A \setminus \xi \), and it is clear that

2. \( H(\rho, \xi) \) is closed in \( H_n \), and

3. \( \dim H(\rho, \xi) = 1 \)

because \( H(\rho, \xi) \) is homeomorphic to \( H_0(A \setminus \xi) \) which has covering dimension one by the result of E. Pol cited above. Now let us put \( H(\rho) = \bigcup \{ H(\rho, \xi) : \xi \in A_n \} \). Clearly the collection \( \{ H(\rho, \xi) : \xi \in A_n \} \) is pairwise disjoint. Moreover we prove

4. \( \{ H(\rho, \xi) : \xi \in A_n \} \) is a discrete collection in \( H(\rho) \). Indeed, if \( x \in H(\rho) \) and \( \rho = (r_1, \ldots, r_n) \), then \( x \in H(\rho, \xi) \) for some \( \xi = (\alpha_1, \ldots, \alpha_n) \in A_n \), i.e. \( x(\alpha_i) = r_i \) for \( i = 1, \ldots, n \). Applying (1), we can take a \( \delta > 0 \) such that

5. \( \delta < |x(\alpha) - r_i| \) for each \( \alpha \in A \setminus \xi \) and \( i = 1, \ldots, n \) because \( x(\alpha) \in P_0 \) for each \( \alpha \in A \setminus \xi \). Let \( x' \) be a point of \( B_\delta(x) \cap H(\rho) \); then there exists \( \xi' = (\alpha'_1, \ldots, \alpha'_n) \in A_n \) such that \( x' \in H(\rho, \xi') \), and hence \( x'(\alpha'_i) = r_i \) for every \( i \). Since \( |x(\alpha) - x'(\alpha)| \leq d(x, x') < \delta \), it follows from (5) that \( x'(\alpha) \notin \{ r_1, \ldots, r_n \} \) and hence \( x'(\alpha) \in P_0 \) for every \( \alpha \in A \setminus \xi \). Therefore we have \( \xi' = \xi \) and \( B_\delta(x) \cap H(\rho) \subseteq H(\rho, \xi) \), which proves (4). From (2), (3) and (4), we obtain

6. \( \dim H(\rho) = 1 \) for every \( \rho \in Q_0^n \).

Since \( Q_0^n \) is countable, to prove Lemma 1 it suffices to show

7. \( H(\rho) \) is closed in \( H_n \) for every \( \rho \).

Let \( x \) be a point in \( H_n \setminus H(\rho) \) and \( \rho = (r_1, \ldots, r_n) \); then there exist \( \xi = (\alpha_1, \ldots, \alpha_n) \in A_n \) and \( \sigma = (s_1, \ldots, s_n) \in Q_0^n \) such that \( x \in H(\sigma, \xi) \). As \( x \notin H(\rho) \), we have \( \rho \neq \sigma \) and \( r_k \neq s_k \) for some \( k \). Applying (1) again, we can choose a \( \delta > 0 \) satisfying

8. \( \delta < |r_k - s_k| \), and

9. \( \delta < |r_i - x(\alpha)| \) for each \( i \) and \( \alpha \in A \setminus \xi \).

To prove \( B_\delta(x) \cap H(\rho) = \emptyset \), assume the contrary, i.e. \( x' \in B_\delta(x) \cap H(\rho) \). Since \( d(x, x') < \delta \), it follows from (9) that \( x'(\alpha) \notin \{ r_1, \ldots, r_n \} \) and hence \( x'(\alpha) \in P_0 \) for every \( \alpha \in A \setminus \xi \), so that \( x'(\alpha_i) = r_i, i = 1, \ldots, n \). Then we have \( |r_k - s_k| = |x'(\alpha_k) - x(\alpha_k)| \leq d(x, x') < \delta \) which contradicts (8), and hence (7) is proved. This completes the proof of Lemma 1.

Since the space \( H_\infty(A) \) can be represented as the union of subspaces \( H_n = H_n(A), n = 1, 2, \ldots \), we see that \( H_\infty(A) \) is countable dimensional by virtue of Lemma 1 and the weight of \( H_\infty(A) \) does not exceed the weight of \( H(A) = |A| \). Therefore to prove Theorem it suffices to show the following Lemma 2, since the space \( K_\infty(A) \), as cited in 1(b), is a universal space for countable dimensional metric spaces with weight \( \leq |A| \).
LEMMA 2. The space $K_\infty(A)$ can be topologically embedded as a subspace of $H_\infty(A)$.

PROOF. Let $S'(A)$ be the subspace of $H(A)$ defined by

$$S'(A) = \{x \in H(A) : |s(x)| \leq 1 \text{ and } 0 < x(\alpha) \leq 1 \text{ if } \alpha \in s(x)\},$$

where $s(x) = \{\alpha \in A : x(\alpha) \neq 0\}$. It is easy to see that $S'(A)$ is homeomorphic to the star space $S(A)$ defined in [5, p. 184]. Since $A$ is infinite we can choose a sequence $\{A_i\}$ of pairwise disjoint sets such that $A = \bigcup A_i$ and $|A_i| = |A|$ for all $i$. Then $S'(A_i)$ is also homeomorphic to the star space $S(A)$. Let

$$K'(A) = \{x \in H(A) : |s(x) \cap A_i| \leq 1 \text{ and } 0 < x(\alpha) \leq 1/i$$

if $\alpha \in s(x) \cap A_i, \ i = 1, 2, \ldots\},$$

and define a bijection $f : K'(A) \to P(A) = \prod_i S'(A_i)$ by

$$(p_i(f(x)))(\alpha) = i \cdot x(\alpha) \quad \text{for each } i \text{ and } \alpha \in A_i,$$

where $p_i$ denotes the projection of $P(A)$ onto $S'(A_i)$. As observed in [2, Theorem 2], $f$ is a homeomorphism. Then the subspace $K'_\infty$ of $K'(A)$ defined by

$$K'_\infty = \{x \in K'(A) : \{|\alpha \in A : x(\alpha) \in Q_0|\} < \aleph_0\}$$

is contained in $H_\infty(A)$, and $f(K'_\infty) = \{y \in P(A) : \{|\alpha \in A : (p_i(y))(\alpha) \in Q_0|\} < \aleph_0\}$ which is homeomorphic to the space $K_\infty(A)$. Hence $K_\infty(A)$ is homeomorphic to $K'_\infty \subseteq H_\infty(A)$, which completes the proof of Lemma 2. Hence the proof of Theorem is also completed.

REFERENCES


DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION, SAITAMA UNIVERSITY, URAWA, JAPAN