NEIGHBORHOODS OF POINTS IN CODIMENSION-ONE SUBMANIFOLDS LIE IN CODIMENSION-ONE SPHERES

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(Communicated by Doug W. Curtis)

ABSTRACT. For \( n \geq 4 \), let \( M \) be an \( (n-1) \)-manifold embedded in an \( n \)-
manifold \( N \). For each point \( p \) of \( M \), there is an \( (n-1) \)-sphere \( \Sigma \) in \( N \) such
that \( \Sigma \cap M \) is a neighborhood of \( p \) in \( M \).

We work in the category of topological manifolds without boundary and topological embeddings.

The \( n = 3 \) case of this result is established by Theorem 5 of [2]. A weak version
of this result for \( n \geq 5 \) is found in Theorem 5B.10 of [3]. This particular proof
arose in response to a private query from D. L. Loveland of Utah State University.

This type of result is used to generalize theorems concerning local properties of
wild codimension-one spheres into theorems about arbitrary wild codimension-one
submanifolds. One application is found in Theorem 6 of [2]. Another application
is discussed on pages 38 and 39 of [1].

PROOF. Without loss of generality, we can cut \( M \) and \( A \) down to assure that
both are orientable. This makes any embedding of \( M \) in \( A \) 2-sided. Now, for an
open subset \( V \) of \( M \), an embedding \( e: V \to A \) is tame if there is an embedding
\( E: V \times \mathbb{R} \to A \) such that \( E(x, 0) = e(x) \) for each \( x \in V \); the embedding \( E \) is called
a collar of \( e \).

Let \( \{U_i : i \geq 0\} \) be a decreasing sequence of open neighborhoods of \( p \) in \( N \) with
diameters converging to zero. Let \( \{D_i: i \geq 0\} \) be a sequence of \( (n-1) \)-balls in \( M \)
such that for each \( i \geq 0 \), \( \{p\} \cup D_{i+1} \subset \text{int}(D_i) \) and \( D_i \subset U_i \). For \( 0 \leq i < j < \infty \),
let \( A(i,j) = (\text{int}(D_i)) - D_j \) and let \( A(i, \infty) = (\text{int}(D_i)) - \{p\} \).

Let \( \varepsilon_0: M \to N \) denote the given inclusion. Repeated applications of Theorem
2.2 of [1] yields a sequence of embeddings \( \varepsilon_i: M \to N \) which, for each \( i \geq 1 \), satisfy
the following three conditions.

1. \( \varepsilon_i = \varepsilon_{i-1} \) on \( M - A(i-1, i+1) \).
2. \( \varepsilon_i|\text{int}(A(0, i+1)) \) is tame.
3. \( \varepsilon_i(D_j) \subset U_j \) for each \( j \geq 0 \).

It follows that the sequence \( \{\varepsilon_i\} \) converges to an embedding \( f: M \to N \) such
that for each \( i \geq 0 \), \( f = \varepsilon_i \) on \( M - A(i, \infty) \). Consequently, \( f|A(0, \infty) \) is tame.

According to [4], \( f \) cannot have isolated wild points. Hence, \( f|\text{int}(D_0) \) is, in
fact, tame. Thus, \( f|\text{int}(D_0) \) has a collar, which we use to slide \( f \) to an embedding
\( g: M \to N \) such that \( g(\text{int}(D_0)) \cap f(\text{int}(D_0)) = \emptyset \) and \( g = f \) on \( M - \text{int}(D_0) \).

As \( p = f(p) \in f(\text{int}(D_0)) \), there is an \( i \geq 0 \) such that \( g(D) \cap U_i = \emptyset \). Since
\( e_i(D_i) \subset U_i \) and \( e_i = f \) on \( M - \text{int}(D_i) \), then \( e_i(\text{int}(D_0)) \cap g(\text{int}(D_0)) = \emptyset \) and

Received by the editors July 27, 1987.
1980 Mathematics Subject Classification (1985 Revision). Primary 57N35, 57N45.

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$e_i = g$ on $\partial D_0$. So $\Sigma = e_i(D_0) \cup g(D_0)$ is an $(n - 1)$-sphere. Since $e_i = e_0$ on $D_{i+1}$, then $D_{i+1} \subset \Sigma \cap M$. □

REFERENCES