THE ISOMORPHISM QUESTION FOR MODULAR GROUP ALGEBRAS OF METACYCLIC p-GROUPS

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ABSTRACT. Let $F[G]$ be a group algebra of a finite $p$-group $G$ over the field $F = GF(p)$. If $G \cong H$, then clearly $F[G] \cong F[H]$. However, it is not known whether the converse is true. The answer for metacyclic $p$-groups, $p > 3$, is given.

Let $p$ be a prime, $G$—a finite nonabelian $p$-group and $F = GF(p)$—the field with $p$ elements. In this note we study the following isomorphism problem: Whether the isomorphism of group algebras $F[G]$ and $F[H]$ implies the isomorphism of groups $G$ and $H$. In [4] it was given the positive answer to this question for all $p$-groups of order at most $p^4$. Here we prove the following

**THEOREM.** If $G$ is a metacyclic $p$-group, $p > 3$, and $F[G] \cong F[H]$, then $G \cong H$.

Our result depends on some lemmas. We use standard notation, see [3, 5].

**LEMMA 1.** Let $N$ be a normal subgroup of a finite $p$-group $G$. If $I = \omega(F[N])F[G]$, then $I^n = \omega(F[N])^nF[G]$. Moreover, if $\eta_i, i = 1, \ldots, t$, form a basis of the space $\omega(F[N])^n$ and $\{1 = g_1, \ldots, g_s\}$ is a right transversal for $N$ in $G$, then $\eta_ig_j, i = 1, \ldots, t, j = 1, \ldots, s$, form a basis of the space $I^n$.

The proof of the first part is by an easy induction on $n$. The second part is obvious.

For any finite $p$-group $G$ let $\{M_i(G)\}$ be the Brauer-Jennings-Zassenhaus $M$-series of a $p$-group $G$ defined by $M_1(G) = G$ and for $n \geq 2$

$$M_n(G) = (M_{n-1}(G), G) M_i(G)^{p^i},$$

where $i$ is the smallest integer satisfying $ip \geq n$.

**LEMMA 2.** If $G'$ is the commutator subgroup of $G$, then the factor groups $M_i(G')/M_{i+1}(G')$ for all $i$ are determined by $F[G]$.

**PROOF.** By proof of [5, 14.2.7] the factor groups $M_i(G')/M_{i+1}(G')$ are determined up to isomorphism by

$$f_j = \dim_F \omega(F[G'])^j/\omega(F[G'])^{j+1} = \frac{1}{|G: G'|} \dim_F \omega(F[G'])^j F[G]/\omega(F[G'])^{j+1} F[G].$$

Since $\omega(F[G'])F[G]$ as an ideal generated by the subspace $[F[G], F[G]]$, is determined by $F[G]$ and $|G: G'| = \dim_F F[G]/\omega(F[G'])F[G]$ the result follows from Lemma 1.

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By the proof of [5, 14.2.7ii] we obtain then

**COROLLARY 1.** If \( G' \) is cyclic, then the isomorphism class of \( G' \) is determined by \( F[G] \). In particular, if \( F[G] \cong F[H] \) and \( G' \) is cyclic, then \( H' \) is cyclic too.

Remark now that if \( G \) is a metacyclic \( p \)-group, then by the inclusion \( (\mathcal{M}_n(G), G) \subset \mathcal{M}_n(G)^{(p)} \) we have \( \mathcal{M}_2(G) = G^{(p)} = \mathcal{M}_3(G) = \cdots = \mathcal{M}_p(G) = \cdots = \mathcal{M}_{p+1}(G) = \cdots = \mathcal{M}_{p^2}(G) = G^{(p^2)} \) and so on. Thus we have

**COROLLARY 2.** If \( G \) is a metacyclic \( p \)-group then the exponent of \( G \) is determined by \( F[G] \).

**LEMMA 3.** If \( G \) is a finite \( p \)-group with \( G' \) cyclic, then for every integer \( n \geq 1 \)

\[
\omega(F[G']^{(p^n)})F[G] = (\omega(F[G'])F[G])^{p^n}.
\]

**PROOF.** Let \( G' = \langle g \rangle \). Elements \( g - 1, (g - 1)^2, \ldots, (g - 1)^{\omega(g) - 1} \) form a basis of \( \omega(F[G']) \) [5, 3.3.3]. So \( \omega(F[G'])^{p^n} \subset \omega(F[G']^{(p^n)})F[G] \subset \omega(F[G'])^{p^n}F[G] \) and then by Lemma 1 \( \omega(F[G']^{(p^n)})F[G] = (\omega(F[G'])F[G])^{p^n} \).

Let now \( \Omega_1(G) = \{ x \in G | x^{p^2} = 1 \} \). We shall say that \( G \) is a \( p_1 \)-group iff \( x^{p^2} = 1 \) implies \( (xy^{-1})^{p^2} = 1 \) for all \( x, y \in G \). Let \( \omega_1(F[G]) \) be the ideal of \( F[G] \) generated by all elements \( a \in F[G] \) satisfying \( a^{p^2} = 0 \).

**LEMMA 4** [1, LEMMA 3]. If \( G \) is a \( p_1 \)-group and for all \( g \in G \) \( g^{p^2} \in Z(G) \), then \( \omega_1(F[G]) = \omega(F[H])F[G] \), where \( H = \Omega_1(G) \).

**PROOF OF THE THEOREM.** Let \( F[G] \cong F[H] \), where \( G \) is metacyclic \( p \)-group and \( p > 3 \). First we show that \( H \) is metacyclic. Let \( \overline{G} \) and \( \overline{H} \) be the factor groups \( G/G' \) and \( H/H' \) respectively. Since by Lemma 3 \( (\omega(F[G'])F[G])^{p} = \omega(F[G']^{(p)})F[G] \) we have

\[
F[\overline{G}] \cong F[G]/(\omega(F[G'])F[G])^{p} \cong F[H]/(\omega(F[H'])F[H])^{p} \cong F[\overline{H}]
\]

The group \( \overline{G} \) satisfies the assumption of Lemma 4, so

\[
\omega_1(F[\overline{G}]) = \omega(F[\Omega_1(\overline{G})])F[\overline{G}].
\]

By the obvious inclusion \( \omega(F[\Omega_1(\overline{H})])F[\overline{H}] \subset \omega_1(F[\overline{H}]) \) we have then

\[
|\overline{H}/\Omega_1(\overline{H})| = \dim_F F[\overline{H}]/\Omega_1(\overline{H}) = \dim_F F[H] - \dim_F \omega(F[\Omega_1(\overline{H})])F[\overline{H}]
\]

\[
\geq \dim_F F[H] - \dim_F \omega(F[H]) = \dim_F F[\overline{G}] - \dim_F \omega_1(F[\overline{G}])
\]

which implies \( |\Omega_1(\overline{H})| \leq |\Omega_1(\overline{G})| = p^2 \). Hence, since \( \overline{H} \) is nonabelian and \( p > 2 \), \( |\Omega_1(\overline{H})| = p^2 \). Now, by [3, III.11.6] \( \overline{H} \) is metacyclic. But we have assumed that \( p > 3 \) and from Corollary 1 \( H' \) is cyclic, that is \( H^{(p)} = \Phi(H')^{\gamma_3}(H) \). So by [3, III.11.3] \( H \) is metacyclic too. The fact that \( H \) is metacyclic and more, that \( G \cong H \), one can obtain also from [6, Theorem 6.25].

Now let \( G \) be generated by elements \( x, y \) with defining relations

(1) \( x^{p^m} = 1, \quad y^{p^n} = x^k, \quad y^{-1}xy = x^r \)

with suitable integers \( m, n, k \) and \( r \) [3, III.11.2]. The isomorphism of \( G \) and \( H \) we prove by induction on the order of \( \langle x \rangle \cap \langle y \rangle \). So assume first that the generators \( x, y \) of \( G \) satisfy

(2) \( x^{p^m} = 1, \quad y^{p^n} = 1, \quad y^{-1}xy = x^r \),
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that is \( \langle x \rangle \cap \langle y \rangle = 1 \). We proceed by induction on the order of \( G' \). The case
\(|G'| = 1\) is known [5, 14.2.7ii]. Since \( H \) is metacyclic the isomorphism of \( G \) and \( H \)
for \(|G'| = p\) follows immediately from [1, Lemma 4 and Corollary of Theorem 2].
This is also the special case of [6, Theorem 6.25]. Let us assume now that \(|G'| = p^s\),
where \( s > 1 \). We have \( \Omega_1(G') = G'(p^{s-1}) \), so by Lemma 3
\[ F[G/\Omega_1(G')] \simeq F[G]/(\omega(F[G'])F[G])p^{s-1} \]
\[ \simeq F[H]/(\omega(F[H'])F[H])p^{s-1} \simeq F[H/\Omega_1(H')] \]
and by the induction \( G/\Omega_1(G') \simeq H/\Omega_1(H') \). Since \( G/\Omega_1(G') \) is generated by
\( \bar{x} = x\Omega_1(G') \), \( \bar{y} = y\Omega_1(G') \) with relations
\[ \bar{x}p^{m-1} = 1, \quad \bar{y}p^n = 1, \quad \bar{y}^{-1}\bar{x}\bar{y} = \bar{x}^r \]
there exist generators \( u, v \) of \( H \) such that
\[ u^{p^m-1}, v^{p^n} \in \Omega_1(H'), \quad v^{-1}uv \equiv u^r \pmod{\Omega_1(H')} \]
Let \( h \in \Omega_1(H') \) be such an element that \( v^{-1}uv = u^rh \), and suppose \( \Omega_1(H') \) is
not a subgroup of \( \langle u \rangle \). Then \( (u, v) = u^{-1}v^{-1}uv = u^{r-1}h \) is a generator of \( H' \)
and \( (u, v)^p = (u^{r-1}h)^p = u^{p(r-1)} \). But \( H' \) is cyclic and \(|H'| > p \) so that we have
\( \Omega_1(H') \leq \langle (u, v)^p \rangle = \langle u^{p(r-1)} \rangle \leq \langle u \rangle \). A contradiction. Thus \( \Omega_1(H') = \langle u^{p^m-1} \rangle \)
and \( v^{p^n} \in \langle u^{p^m-1} \rangle, v^{-1}uv = u^{r+tp^{m-1}} \) for suitable \( t \). Suppose that \( v^{p^n} \neq 1 \). If
\( n > m \), then \( \exp(G) = p^n < p^{n+1} = \exp(H) \) which by Corollary 2 is impossible.
If \( n < m \) then one can choose an element \( u_1 \) of \( \langle u \rangle \) such that \( u_1^{p^n} = v^{p^n} \). Now
replacing \( v \) by \( v_1 = v^{u_1^{-1}} \) we have \( H = \langle u, v_1 \rangle, v_1^{p^n} = 1 \) and \( v_1^{-1}uv = v^{-1}uv \). The
isomorphism of \( G \) and \( H \) follows now from [2, Lemma 8].
Suppose now that \( x \) and \( y \) are generators of \( G \) with the defining relations (1), the
possible smallest order of \( \langle x \rangle \cap \langle y \rangle \) and \( \langle x \rangle \cap \langle y \rangle \neq 1 \). Remark that the order of \( y \)
must be greater than the order of \( x \). Otherwise, using standard considerations one
can replace \( y \) by an element \( y_1 \) such that \( \langle x, y_1 \rangle = G \) and \( \langle x \rangle \cap \langle y_1 \rangle = 1 \). Assuming
\( G/\Omega_1(G') \simeq H/\Omega_1(H') \) as above we can choose generators \( u \) and \( v \) of \( H \) such that
\( \Omega_1(H') \leq \langle u \rangle \) and
\[ u^{p^m} = 1, \quad v^{-1}uv = u^{r+tp^{m-1}}, \quad v^{p^n} = u^{p^k+sp^{m-1}} \]
for suitable integers \( s, t \). If \( k < m - 1 \), then \( p^k + sp^{m-1} \neq 0 \pmod{p^m} \) and there
exist an element \( u_1 \in \langle u \rangle \) such that \( \langle u \rangle = \langle u_1 \rangle, v^{p^n} = u_1^{p^k} \) and \( v^{-1}u_1v = v^{r+tp^{m-1}} \).
So by [2, Lemma 8] \( G \simeq H \). If \( p^k + sp^{m-1} \equiv 0 \pmod{p^m} \), then \( v^{p^n} = 1 \) which
implies \( \exp(H) = p^n < \exp G \). By Corollary 2 it is impossible. This completes the
proof.

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