THE ISOMORPHISM QUESTION FOR MODULAR GROUP
ALGEBRAS OF METACYCLIC p-GROUPS

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ABSTRACT. Let \( F[G] \) be a group algebra of a finite p-group \( G \) over the field \( F = GF(p) \). If \( G \simeq H \), then clearly \( F[G] \simeq F[H] \). However, it is not known whether the converse is true. The answer for metacyclic p-groups, \( p > 3 \), is given.

Let \( p \) be a prime, \( G \)—a finite nonabelian p-group and \( F = GF(p) \)—the field with \( p \) elements. In this note we study the following isomorphism problem: Whether the isomorphism of group algebras \( F[G] \) and \( F[H] \) implies the isomorphism of groups \( G \) and \( H \). In [4] it was given the positive answer to this question for all p-groups of order at most \( p^4 \). Here we prove the following

**THEOREM.** If \( G \) is a metacyclic p-group, \( p > 3 \), and \( F[G] \simeq F[H] \), then \( G \simeq H \).

Our result depends on some lemmas. We use standard notation, see [3, 5].

**LEMMA 1.** Let \( N \) be a normal subgroup of a finite p-group \( G \). If \( I = \langle \omega(F[N])F[G] \rangle \), then \( I^n = \langle \omega(F[N])^nF[G] \rangle \). Moreover, if \( \eta_i, i = 1, \ldots, t \), form a basis of the space \( \omega(F[N]) \) and \( \{1 = g_1, \ldots, g_s\} \) is a right transversal for \( N \) in \( G \), then \( \eta_i g_j, i = 1, \ldots, t, j = 1, \ldots, s \), form a basis of the space \( I^n \).

The proof of the first part is by an easy induction on \( n \). The second part is obvious.

For any finite p-group \( G \) let \( \{\mathcal{M}_i(G)\} \) be the Brauer-Jennings-Zassenhaus \( \mathcal{M} \)-series of a p-group \( G \) defined by \( \mathcal{M}_1(G) = G \) and for \( n \geq 2 \)

\[
\mathcal{M}_n(G) = (\mathcal{M}_{n-1}(G), G)\mathcal{M}_1(G)^{(p)}
\]

where \( i \) is the smallest integer satisfying \( ip \geq n \).

**LEMMA 2.** If \( G' \) is the commutator subgroup of \( G \), then the factor groups \( \mathcal{M}_i(G')/\mathcal{M}_{i+1}(G') \) for all \( i \) are determined by \( F[G] \).

**PROOF.** By proof of [5, 14.2.7i] the factor groups \( \mathcal{M}_i(G')/\mathcal{M}_{i+1}(G') \) are determined up to isomorphism by

\[
f_j = \dim_F \omega(F[G'])^j/\omega(F[G'])^{j+1}
\]

\[
= \frac{1}{|G: G'|} \dim_F \omega(F[G'])^j F[G]/\omega(F[G'])^{j+1} F[G].
\]

Since \( \omega(F[G'])F[G] \) as an ideal generated by the subspace \( [F[G], F[G]] \), is determined by \( F[G] \) and \( |G: G'| = \dim_F F[G]/\omega(F[G'])F[G] \) the result follows from Lemma 1.

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By the proof of [5, 14.2.7.i] we obtain then

**Corollary 1.** If $G'$ is cyclic, then the isomorphism class of $G'$ is determined by $F[G]$. In particular, if $F[G] \simeq F[H]$ and $G'$ is cyclic, then $H'$ is cyclic too.

Remark now that if $G$ is a metacyclic $p$-group, then by the inclusion $(\mathcal{M}_n(G), G) \subseteq \mathcal{M}_n(G)^{(p)}$ we have $\mathcal{M}_n(G) = G^{(p)} = \cdots = \mathcal{M}_p(G), \mathcal{M}_{p+1}(G) = \cdots = \mathcal{M}_{p^2}(G) = G^{(p^2)}$ and so on. Thus we have

**Corollary 2.** If $G$ is a metacyclic $p$-group then the exponent of $G$ is determined by $F[G]$.

**Lemma 3.** If $G$ is a finite $p$-group with $G'$ cyclic, then for every integer $n \geq 1$

$$\omega(F[G'^{(p^n)}])F[G] = (\omega(F[G'])F[G])^{p^n}.$$  

**Proof.** Let $G' = \langle g \rangle$. Elements $g - 1, (g - 1)^2, \ldots, (g - 1)^{\omega(g) - 1}$ form a basis of $\omega(F[G'])$ [5, 3.3.3]. So $\omega(F[G'])^{p^n} \subseteq \omega(F[G'^{(p^n)}])F[G] \subseteq \omega(F[G'])^{p^n}F[G]$ and then by Lemma 1 $\omega(F[G'^{(p^n)}])F[G] = (\omega(F[G'])F[G])^{p^n}$.

Let now $\Omega_1(G) = \{x \in G | x^p = 1\}$. We shall say that $G$ is a $p_1$-group iff $x^p = y^p$ implies $(xy^{-1})^p = 1$ for all $x, y \in G$. Let $\omega_1(F[G'])$ be the ideal of $F[G]$ generated by all elements $a \in F[G]$ satisfying $a^p = 0$.

**Lemma 4 [1, Lemma 3].** If $G$ is a $p_1$-group and for all $g \in G \, g^p \in Z(G)$, then $\omega_1(F[G]) = \omega(F[H])F[G]$, where $H = \Omega_1(G)$.

**Proof of the Theorem.** Let $F[G] \simeq F[H]$, where $G$ is a metacyclic $p$-group and $p > 3$. First we show that $H$ is metacyclic. Let $G$ and $H$ be the factor groups $G/G'^{(p)}, H/H'^{(p)}$ respectively. Since by Lemma 3 $(\omega(F[G'])F[G])^{p^n} = \omega(F[G'^{(p^n)}])F[G]$ we have

$$F[G] \simeq F[G]/(\omega(F[G'])F[G])^{p^n} \simeq F[H]/(\omega(F[H'])F[H])^{p^n} \simeq F[H].$$

The group $\overline{G}$ satisfies the assumption of Lemma 4, so

$$\omega_1(F[\overline{G}]) = \omega(F[\Omega_1(\overline{G})])F[\overline{G}].$$

By the obvious inclusion $\omega(F[\Omega_1(H)])F[H] \subseteq \omega_1(F[H])$ we have then

$$|H/\Omega_1(H)| = \dim_F F[H/\Omega_1(H)] = \dim_F F[H] - \dim_F \omega(F[\Omega_1(H)])F[H]$$

$$\geq \dim_F F[H] - \dim_F \omega_1(F[H]) = \dim_F F[\overline{G}] - \dim_F \omega_1(F[\overline{G}])$$

which implies $|\Omega_1(H)| \leq |\Omega_1(\overline{G})| = p^2$. Hence, since $\overline{H}$ is nonabelian and $p > 2, |\Omega_1(\overline{H})| = p^2$. Now, by [3, III.11.6] $\overline{H}$ is metacyclic. But we have assumed that $p > 3$ and from Corollary 1 $H'$ is cyclic, that is $H'^{(p)} = \Phi(H')\gamma_3(H)$. So by [3, III.11.3] $H$ is metacyclic too. The fact that $\overline{H}$ is metacyclic and more, that $\overline{G} \simeq \overline{H}$, one can obtain also from [6, Theorem 6.25].

Now let $G$ be generated by elements $x, y$ with defining relations

$$x^{p^n} = 1, \quad y^{p^n} = x^{p^k}, \quad y^{-1}xy = x^r$$

with suitable integers $m, n, k$ and $r$ [3, III.11.2]. The isomorphism of $G$ and $H$ we prove by induction on the order of $\langle x \rangle \cap \langle y \rangle$. So assume first that the generators $x, y$ of $G$ satisfy

$$x^{p^n} = 1, \quad y^{p^n} = 1, \quad y^{-1}xy = x^r,$$
that is \( \langle x \rangle \cap \langle y \rangle = 1 \). We proceed by induction on the order of \( G' \). The case
\( |G'| = 1 \) is known [5, 14.2.7ii]. Since \( H \) is metacyclic the isomorphism of \( G \) and \( H \)
for \( |G'| = p \) follows immediately from [1, Lemma 4 and Corollary of Theorem 2].
This is also the special case of [6, Theorem 6.25]. Let us assume now that \( |G'| = p^s \),
where \( s > 1 \). We have \( \Omega_1(G') = G'(p^{s-1}) \), so by Lemma 3
\[
F[G/\Omega_1(G')] \simeq F[G]/(\omega(F[G'])F[G])p^{s-1}
\]
and by the induction \( G/\Omega_1(G') \simeq H/\Omega_1(H') \). Since \( G/\Omega_1(G') \) is generated by
\[ \bar{x} = x\Omega_1(G'), \quad \bar{y} = y\Omega_1(G') \]
there exist generators \( u, v \) of \( H \) such that
\[ u^{p^m}, v^{p^n} \in \Omega_1(H'), \quad v^{-1}uv \equiv u^\tau \pmod{\Omega_1(H')} \]
Let \( h \in \Omega_1(H') \) be such an element that \( v^{-1}uv = u^\tau h \), and suppose \( \Omega_1(H') \)
is not a subgroup of \( \langle u \rangle \). Then \( \langle u, v \rangle = u^{-1}v^{-1}uv = u^{-1}h \) is a generator of \( H' \)
and \( (u, v)^p = (u^{p-1}h)^p = u^{p(r-1)} \). But \( H' \) is cyclic and \( |H'| > p \) so that we have
\( \Omega_1(H') \leq \langle (u,v)^p \rangle = \langle u^{p(r-1)} \rangle \leq \langle u \rangle \). A contradiction. Thus \( \Omega_1(H') = \langle u^{p^{m-1}} \rangle \)
and \( v^{p^n} \in \langle u^{p^{m-1}} \rangle \), \( v^{-1}uv = u^{r+tp^{m-1}} \) for suitable \( t \). Suppose that \( v^{p^n} \neq 1 \). If
\( n > m \), then \( \exp(G) = p^n < p^{n+1} = \exp(H) \) which by Corollary 2 is impossible.
If \( n < m \) then one can choose an element \( u_1 \) of \( \langle u \rangle \) such that \( u_1^{p^n} = v^{p^n} \). Now
replacing \( v \) by \( v_1 = v u_1^{-1} \) we have \( H = \langle u, v_1 \rangle \), \( v_1^{p^n} = 1 \) and \( v_1^{-1}uv_1 = v_1 = v^{-1}uv \). The
isomorphism of \( G \) and \( H \) follows now from [2, Lemma 8].
Suppose now that \( x \) and \( y \) are generators of \( G \) with the defining relations (1), the
possible smallest order of \( \langle x \rangle \cap \langle y \rangle \) and \( \langle x \rangle \cap \langle y \rangle \neq 1 \). Remark that the order of \( y \)
must be greater than the order of \( x \). Otherwise, using standard considerations one
can replace \( y \) by an element \( y_1 \) such that \( \langle x, y_1 \rangle = G \) and \( \langle x \rangle \cap \langle y_1 \rangle = 1 \). Assuming
\( G/\Omega_1(G') \simeq H/\Omega_1(H') \) as above we can choose generators \( u \) and \( v \) of \( H \) such that
\( \Omega_1(H') \leq \langle u \rangle \) and
\[ u^{p^m} = 1, \quad v^{-1}uv = u^{r+tp^{m-1}}, \quad v^{p^n} = u^{p^k+sp^{m-1}} \]
for suitable integers \( s, t \). If \( k < m - 1 \), then \( p^k + sp^{m-1} \neq 0 \pmod{p^m} \) and there
exist an element \( u_1 \in \langle u \rangle \) such that \( \langle u \rangle = \langle u_1 \rangle \), \( v^{p^n} = u_1^{p^k} \) and \( v^{-1}u_1v = u_1^{r+tp^{m-1}} \).
So by [2, Lemma 8] \( G \simeq H \). If \( p^k + sp^{m-1} \equiv 0 \pmod{p^m} \), then \( v^{p^n} = 1 \) which
implies \( \exp(H) = p^n < \exp G \). By Corollary 2 it is impossible. This completes the proof.
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