THE HODGE GROUP OF AN ABELIAN VARIETY

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ABSTRACT. Let $A$ be a simple abelian variety of odd dimension, defined over $\mathbb{C}$. If the Hodge classes on $A$ are intersections of divisors, then the semisimple part of the Hodge group of $A$ is as large as it is allowed to be by endomorphisms and polarizations.

1. Introduction. Let $A$ be an abelian variety defined over $\mathbb{C}$, and denote by $\mathcal{H}(A)$ its Hodge ring:

$$\mathcal{H}(A) = \bigoplus_p (H^{2p}(A(\mathbb{C}), \mathbb{Q}) \cap H^{pp}).$$

Denote by $\mathcal{D}(A)$ the subring of $\mathcal{H}(A)$ generated by the elements of $H^{2}(A(\mathbb{C}), \mathbb{Q}) \cap H^{11}$. We denote by $\text{Hod}(A)$ the Hodge group of $A$ introduced by Mumford [3]. It is a connected, reductive algebraic subgroup of $GL(V)$, where $V = H_1(A(\mathbb{C}), \mathbb{Q})$. It acts on the cohomology of $A$ and its main property is that

$$H^*(A(\mathbb{C}), \mathbb{Q})^{\text{Hod}(A)} = \mathcal{H}(A).$$

We shall also consider a group $L(A)$ which was studied by Ribet [8] and the author [6]. To define it, we fix a polarization $\psi$ of $A$. Then, $L(A)$ is the connected component of the identity of the centralizer of $\text{End}(A) \otimes \mathbb{Q}$ in $Sp(V, \psi)$. The definition is independent of the choice of polarization. It is known that $\text{Hod}(A) \subseteq L(A)$ and in [6], it was proved that if $A$ has no simple factor of type III (see §2 for the definitions), then

$$\text{Hod}(A) = L(A) \Leftrightarrow \mathcal{H}(A^k) = \mathcal{D}(A^k) \text{ for all } k \geq 1.$$ 

Moreover, if $A$ has a factor of type III, then $\mathcal{H}(A) \neq \mathcal{D}(A)$.

Let $G$ be a connected reductive algebraic group defined over $\mathbb{Q}$, and denote by $\mathcal{G}$ its Lie algebra. It is known that there is a unique connected semisimple algebraic subgroup $G_{ss}$ of $G$ whose Lie algebra is the maximal semisimple subalgebra of $\mathcal{G}$. We refer to $G_{ss}$ as 'the semisimple part' of $G$.

The purpose of this note is to show that in some cases, the assumption $\mathcal{H}(A) = \mathcal{D}(A)$ is already sufficient to imply that the semisimple parts of $\text{Hod}(A)$ and $L(A)$ are equal. We prove two general results of which the following is a consequence.
PROPOSITION. Let $A$ be simple and of odd dimension. Then, $\mathcal{H}(A) = \mathcal{D}(A)$ implies that $\text{Hod}(A)_{\text{ss}} = L(A)_{\text{ss}}$.

A key element in the proof is the classification (cf. Serre [10]) of the simple factors of $\text{Hod}(A)$ and the explicit determination of $L(A)$ [6].

REMARKS. 1. By a classical result of Lefschetz, $\mathcal{D}(A)$ consists of Poincaré duals of algebraic cycles on $A$. Thus, the condition $\mathcal{H}(A) = \mathcal{D}(A)$ implies that the Hodge conjecture holds for $A$. (In fact, it holds in the strong sense that any algebraic cycle on $A$ is algebraically equivalent to an intersection of divisors.)

2. The proof will show that the condition $\mathcal{H}(A) = \mathcal{D}(A)$ can be replaced by the weaker condition that every Hodge cycle on $A$ of codimension $\leq 3$ is contained in $\mathcal{D}(A)$.

3. If we are in a case where $L(A)$ is known to be semisimple, the Proposition shows that $\mathcal{H}(A) = \mathcal{D}(A)$ implies that $\mathcal{H}(A_k) = \mathcal{D}(A_k)$ for all $k \geq 1$. This is because the conditions on $A$ preclude it from being of type III and so (*) can be applied.

4. If $A$ is simple and of CM type, both $\text{Hod}(A)$ and $L(A)$ are tori, and so the Proposition above gives no information. However, Lenstra and Ribet (unpublished) have shown in this case that if $\mathcal{H}(A) = \mathcal{D}(A)$ and $\text{End}(A) \otimes \mathbb{Q}$ is a field which is abelian over $\mathbb{Q}$, then $\text{Hod}(A) = L(A)$. It is an interesting problem to study how small $\text{Hod}(A)$ can be, relative to $L(A)$, when $A$ is of CM type. Liem Mai [12] has recently obtained lower bounds for the dimension of $\text{Hod}(A)$, in some cases, and Dodson [1] has shown that this dimension must satisfy some congruence conditions.

5. Tanke'ev has shown that if $A$ is simple and of prime dimension, $\text{Hod}(A) = L(A)$ and $\mathcal{H}(A_k) = \mathcal{D}(A_k)$ for all $k \geq 1$. The proof has been simplified by Ribet [8] who also showed that the conclusion is valid for a slightly larger class of abelian varieties.

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2. Lemmas. Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let $\rho$ denote complex conjugation. We write

$$\Delta(A) = \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$$

and let $E$ be a maximal commutative semisimple subalgebra of $\Delta(A)$. Thus, $E$ is a product of fields $E = E_1 \times \cdots \times E_r$ and by the classification of Albert (cf. Mumford [4, §21]) each factor is either a CM field or a totally real field. We say that $A$ is of type III if $\Delta(A)$ is a totally definite quaternion division algebra over a totally real field, and $A$ is of type IV if $\Delta(A)$ is a division algebra over a CM field.

We have a decomposition $V \otimes \mathbb{C} = \prod V_{\sigma}$ indexed by the set $\Sigma$ of homomorphisms $\sigma: E \otimes \mathbb{C} \rightarrow \mathbb{C}$, where

$$V_{\sigma} = (V \otimes \mathbb{C}) \otimes_{E \otimes \mathbb{C}, \sigma} \mathbb{C}.$$

Let $H = \text{Hod}(A)/\mathbb{C}$ and $L = L(A)/\mathbb{C}$. Each $V_{\sigma}$ is an $L$ module and hence, also an $H$ module. Denote by $L_{\sigma}$ the projection of $L$ to $GL(V_{\sigma})$. 

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LEMMA 1. Each $V_\sigma$ is a simple $H$ module, and the following are equivalent:
(i) $V_\sigma_1 \sim V_\sigma_2$ as $H$ modules,
(ii) $V_\sigma_1 \sim V_\sigma_2$ as $L$ modules.

PROOF. It follows easily from the definitions that
\[ \text{End}_{H(A)}(V) = \text{End}_{L(A)} V = \Delta(A). \]
As $E$ is a maximal commutative semisimple subalgebra, we have
\[ \text{End}_{H(A)}, E V = E. \]
Complexifying, we see that for any $\sigma \in \Sigma$, we have
\[ \text{End}_H V_\sigma = C. \]
This proves the first assertion. To see the equivalence of (i) and (ii), note that for any $a_1, a_2 \in E$, we have $\text{Hom}_L(V_{a_1}, V_{a_2}) \leftrightarrow \text{Hom}_H(V_{a_1}, V_{a_2})$. Now
\[ \prod \text{Hom}_H(V_{a_1}, V_{a_2}) = \prod \text{Hom}_L(V_{a_1}, V_{a_2}) \]
as both sides are equal to $\Delta(A) \otimes C$. (The product on both sides is over all pairs $a_1, a_2 \in \Sigma$.) The result follows. \qed

Denote by $E_i^+$ the maximal totally real subfield of $E_i$ ($1 \leq i \leq r$) and set $E^+ = E_1^+ \times \cdots \times E_r^+$. For each homomorphism $\lambda : E^+ \otimes R \to R$, set
\[ X_\lambda = (V \otimes R) \otimes_{E^+ \otimes R} R. \]
Since $L(A)$ commutes with $E$, it acts on $X_\lambda$. We write $L_\lambda$ for the projection $L(A) \to GL(X_\lambda)$.

Suppose that every simple factor of $A$ is of type IV. Then, for every $\lambda$, $X_\lambda$ has a natural complex structure and the polarization induces a Hermitian form $\omega_\lambda$ on $X_\lambda$. Denote by $U(X_\lambda, \omega_\lambda)$ the unitary group with respect to $\omega_\lambda$.

LEMMA 2. If every simple factor of $A$ is of type IV, then for every $\lambda$, $L_\lambda = U(X_\lambda, \omega_\lambda)$.

PROOF. This follows from Lemma 2.1 and Lemma 2.3 of [6]. \qed

We know that
\[ H \subset L \subset \prod GL(V_\sigma). \]
Let $S = H_{ss}$ and $R = L_{ss}$. Let $H_\sigma$ denote the projection of $H$ to $GL(V_\sigma)$, and $S_\sigma$ the projection of $S$ to $SL(V_\sigma)$.

LEMMA 3. If $V_\sigma$ is of odd dimension (over $C$), then every simple component of $S_\sigma$ is of type $A$. Moreover, if $V_\sigma = V_{1,\sigma} \otimes V_{2,\sigma} \otimes \cdots \otimes V_{m,\sigma}$ is a decomposition of $V_\sigma$ induced by the decomposition of $S_\sigma$ into simple components, each $V_{i,\sigma}$ is an exterior power of the standard representation (i.e. a fundamental representation).

This is [10, Corollary to Proposition 8].

LEMMA 4. Let $S_i$ ($1 \leq i \leq r$) be complex simple Lie groups of type $A_{m(i)}$, $m(i) \geq 3$ for all $i$. Let $Y_1, \ldots, Y_r$ be fundamental representations of $S_1, \ldots, S_r$ (respectively). Let
\[ S = S_1 \times S_2 \times \cdots \times S_r \quad \text{and} \quad Y = Y_1 \otimes Y_2 \otimes \cdots \otimes Y_r. \]
If the \( S \) modules \( \wedge^2 Y \) and \( \wedge^3 Y \) are irreducible, then \( r = 1 \) and \( Y \) is the standard representation of \( S \) or its dual.

**PROOF.** For each \( i \),
\[
\text{Sym}^2(Y_i) \otimes \cdots \otimes \text{Sym}^2(Y_{i-1}) \otimes \wedge^2 Y_i \otimes \text{Sym}^2(Y_{i+1}) \otimes \cdots \otimes \text{Sym}^2(Y_r)
\]
is an \( S \) submodule of \( \wedge^2 Y \). Thus, \( \wedge^2 Y \) is reducible unless \( r = 1 \). Now, let \( W \) denote the standard representation of \( S \) and set \( w = \dim W \). Suppose that \( Y = \wedge^2 W \). We may assume that \( j \leq w/2 \) by replacing \( Y \) with its dual if necessary. In the notation of Jacobson [2, Chapters 4 and 8], the highest weight of the exterior square of \( Y \) is \( \alpha_j + \alpha_{j+1} \). By the Weyl dimension formula, we must then have
\[
\left( \begin{array}{c} w \\ 2 \end{array} \right) = \dim \wedge^2 Y = \frac{3(w+1)}{(j+1)(w-j+2)} \left( \begin{array}{c} w \\ j \end{array} \right) \left( \begin{array}{c} w \\ j+1 \end{array} \right).
\]
Easy estimates show that this is possible only if \( j \leq 2 \). If \( j = 2 \), the highest weight of the exterior cube of \( Y \) is \( 2\alpha_1 + \alpha_4 \) and again, by the dimension formula, we must have
\[
\left( \begin{array}{c} w \\ 3 \end{array} \right) = \dim \wedge^3 Y = \frac{1}{3}(w+1)(w+2) \left( \begin{array}{c} w \\ 4 \end{array} \right)
\]
which forces \( w \leq 2 \). Thus, we must have \( j = 1 \) and the lemma is proved.

**LEMMA 5.** Let \( W_1, W_2 \) be two finite dimensional complex vector spaces. Let \( \mathcal{S}_1, \mathcal{S}_2 \) be simple complex Lie subalgebras of \( \mathfrak{gl}(W_1), \mathfrak{gl}(W_2) \) (respectively) of type \( A, B \) or \( C \). Let \( \mathcal{S} \) be a Lie subalgebra of \( \mathcal{S}_1 \times \mathcal{S}_2 \) whose projection to each factor is surjective. Then, either \( \mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \) or \( \mathcal{S} \) is the graph of an isomorphism of \( \mathcal{S}_1 \approx \mathcal{S}_2 \) induced by an isomorphism \( W_2 \approx W_1 \) or \( W_1 \approx W_2 \) of \( \mathcal{S} \)-modules.

**PROOF.** For \( i = 1, 2 \), let \( \pi_i : \mathcal{S} \to \mathcal{S}_i \) denote the \( i \)th projection and set \( \mathcal{N}_i = \ker \pi_{3-i} \). We may view \( \mathcal{N}_i \) as an ideal of \( \mathcal{S}_i \) and hence, \( \mathcal{N}_i = 0 \). By Goursat's Lemma (see Ribet [7, §5]), it follows that \( \mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \) or \( \mathcal{S} \) is the graph of an isomorphism \( \mathcal{S}_1 \approx \mathcal{S}_2 \). In the latter case, it follows that \( \mathcal{S}_1, \mathcal{S}_2 \) are both of the same type. If they are of type \( B \) or \( C \), it follows that \( W_1 \approx W_2 \) as \( \mathcal{S} \)-modules, since every automorphism of such algebras is inner [2, p. 281]. If they are of type \( A \), then \( W_1 \approx W_2 \) or \( W_2 \approx W_1 \) as \( \mathcal{S} \)-modules, since, modulo inner automorphisms, the only automorphism of such an algebra is \( g \mapsto -g \).

**LEMMA 6.** Let \( I \) be a finite set and for each \( \sigma \in I \), let \( \mathcal{S}_\sigma \) be a finite dimensional complex simple Lie algebra. Let \( \mathcal{A}, \mathcal{B} \) be two algebras such that
(a) \( \mathcal{A} \subseteq \mathcal{B} \).
(b) \( \mathcal{B} \) is a subalgebra of \( \prod \mathcal{S}_\sigma \) and the projection to each \( \mathcal{S}_\sigma \) is surjective.
(c) \( \mathcal{A}, \mathcal{B} \) have equal images on \( \mathcal{S}_\sigma \times \mathcal{S}_\tau \) for all pairs \( (\sigma, \tau) \in I \times I, \sigma \neq \tau \).
Then, \( \mathcal{A} = \mathcal{B} = \prod \mathcal{S}_\sigma \) where the \( \sigma \) range over a certain subset of \( I \).

This is [9, Lemma 4.6].

**3. Main results.** We retain the notation of the previous section. Thus, \( A \) is an abelian variety, \( V = H_1(A(C), \mathbb{Q}) \) and \( E \) is a maximal commutative semisimple subalgebra of \( \Delta(A) \).
THEOREM 1. Suppose that $E$ is a product of CM fields and that $V$ is free over $E$ of odd rank $m$. Suppose that $\mathcal{H}(A) = \mathcal{D}(A)$. Then $\text{Hod}(A)_{ss} = L(A)_{ss}$.

PROOF. First, we check that every simple factor of $A$ is of type IV. We have $E = E_1 \times \cdots \times E_r$, with $E_i$ a CM field. There is a corresponding decomposition up to isogeny $A \sim A_1 \times \cdots \times A_r$. As $E_i$ is a field, $A_i$ is a power of a simple abelian variety, say $A_i = B_i^{n(i)}$. To show that $B_i$ is of type IV, it suffices to show that $\text{End}(B_i) \otimes \mathbb{Q}$ is not a quaternion division algebra $D_i$ over a totally real field $F_i$. Suppose that it is and let $x_i = \dim_D H_1(B_i(C), \mathbb{Q})$. As $E_i$ is a maximal commutative subfield of $A(A_i) = \mathbb{M}_n(i)(D_i)$,

$$\dim V_i = \frac{n(i)x_i[D_i : \mathbb{Q}]}{2n(i)[F_i : \mathbb{Q}]} = 2x_i.$$  

This contradicts our assumption that $m$ is odd. Thus, every simple factor of $A$ is of type IV.

Now, for each homomorphism $\lambda : E^+ \otimes \mathbb{R} \rightarrow \mathbb{R}$, we have

$$X_\lambda \otimes \mathbb{C} = V_\sigma \oplus V_{\rho \sigma}$$

where $\sigma$ is an extension of $\lambda$ to a map $E \otimes \mathbb{C} \rightarrow \mathbb{C}$. Lemma 2 implies that as $L$ modules,

(1) $$V_{\rho \sigma} \simeq \tilde{V}_\sigma.$$  

Moreover, Lemma 1 implies that the $V_\sigma$ are all simple as $H$ modules. Let $R_\sigma$ denote the image of $R$ in $SL(V_\sigma)$. By Lemma 2, $R_\sigma = SL(V_\sigma)$. As $\dim_{\mathbb{C}}(V_\sigma) = m$ is odd, $m = 1$ or $m \geq 3$. In the first case, $S_\sigma = R_\sigma = 1$. We may thus suppose that $m \geq 3$. By Lemma 3, there is an isogeny $S_\sigma \sim S_1,\sigma \times \cdots \times S_r,\sigma$ with each $S_i,\sigma$ a simple group of type $A$. In the corresponding decomposition $V_\sigma = V_{1,\sigma} \otimes \cdots \otimes V_{r,\sigma}$, each $V_i,\sigma$ is a fundamental representation of $S_i,\sigma$.

By [6, Lemmas 2.1, 3.4 and 3.6] and our assumption that $\mathcal{H}(A) = \mathcal{D}(A)$, we see that

$$H^*(A(C), \mathbb{Q})_{\text{Hod}(A)} = \mathcal{H}(A) = \mathcal{D}(A) = H^*(A(C), \mathbb{Q})_{L(A)}.$$  

Thus, for any sequence $\{i(\sigma)\}_\sigma$ of positive integers,

(\*) $$\text{the } H \text{ and } L \text{ invariants of } \bigotimes_\sigma \bigwedge^i V_\sigma \text{ must agree.}$$

In particular, utilising (1), we see that for any $\sigma$, and any integer $i$ with $1 \leq i \leq m$, we must have

(2) $$\dim \text{End}_{H_\sigma} \left( \bigwedge^i V_\sigma \right) = \dim \text{End}_{L_\sigma} \left( \bigwedge^i V_\sigma \right) = 1.$$  

Since $H_\sigma \subseteq L_\sigma = GL(V_\sigma)$, the center $Z_\sigma$ of $H_\sigma$ consists at most of scalars. Thus (2) is equivalent to

(3) $$\text{End}_{S_\sigma} \left( \bigwedge^i V_\sigma \right) = \mathbb{C}$$

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for all $\sigma$. In particular, $\wedge^2 V_\sigma$ and $\wedge^3 V_\sigma$ are irreducible as $S_\sigma$ modules. Since $S_\sigma \subseteq R_\sigma$ and $R_\sigma$ acts on $V_\sigma$ by the standard representation, Lemma 4 implies that
$$S_\sigma = SL(V_\sigma) = R_\sigma.$$ In particular, the Lie algebra $\mathcal{L}_\sigma$ of $S_\sigma$ is $\mathcal{L}L(V_\sigma)$. Now, for $\sigma, \tau \in \Sigma, \sigma \neq \tau$, let $S_{\sigma, \tau}$ (respectively, $R_{\sigma, \tau}$) denote the image of $S$ (respectively, $R$) in $SL(V_\sigma) \times SL(V_\tau)$. Let $\mathcal{L}_{\sigma, \tau}$ (respectively, $\mathcal{R}_{\sigma, \tau}$) denote the Lie algebra of $S_{\sigma, \tau}$ (respectively, $R_{\sigma, \tau}$). Goursat’s Lemma implies that
$$\mathcal{L}_{\sigma, \tau} = \mathcal{L}L(V_\sigma) \times \mathcal{L}L(V_\tau)$$ or $\mathcal{L}_{\sigma, \tau}$ is the graph of an isomorphism
$$\mathcal{R}_{\sigma, \tau} \simeq \mathcal{L}L(V_\tau).$$ If (4) occurs, then we see that
$$\mathcal{S}_{\sigma, \tau} = \mathcal{S}S(V_\sigma) \times \mathcal{S}S(V_\tau)$$ also, since $\mathcal{S}_{\sigma, \tau} \subseteq \mathcal{R}_{\sigma, \tau}$.

Now suppose (5) occurs. Lemma 5 and (1) imply that
$$V_\sigma \simeq V_\tau \text{ or } V_{\rho \sigma}$$ as $S$ modules. If $\tau = \rho \sigma$, then it is clear that $\mathcal{R}_{\sigma, \tau}$ is also the graph of the isomorphism (5). We may therefore suppose that $\tau$ is different from $\rho \sigma$. Consider the $H$ submodule of $H_4(A(C), \mathbb{C})$ given by
$$W = V_\sigma \otimes V_{\rho \sigma} \otimes V_\tau \otimes V_{\rho \tau}.$$ From (6), it follows that as $S$ modules,
$$W \simeq V_\sigma \otimes V_\sigma \otimes V_{\rho \sigma} \otimes V_{\rho \sigma} \simeq \text{End}(V_\sigma \otimes V_\sigma)$$
$$\simeq \text{End} \left( \wedge^2 V_\sigma \oplus \text{Sym}^2 V_\sigma \right).$$ The latter contains a two dimensional subspace
$$W_{\sigma, \tau} = \text{End}_S \left( \wedge^2 V_\sigma \right) \oplus \text{End}_S \left( \text{Sym}^2 V_\sigma \right)$$ on which $S$ acts trivially. The center of $H$ acts trivially on $W$, and hence also on $W_{\sigma, \tau}$. It follows that $H$ acts trivially on $W_{\sigma, \tau}$. By our assumption (**), $L$ also acts trivially on this space. It follows that (6) holds as $L$ modules also, for otherwise, the subspace of $W$ on which $L$ acts trivially is only one dimensional. Hence, we deduce that $\mathcal{R}_{\sigma, \tau}$ is the graph of an isomorphism $\mathcal{L}L(V_\sigma) \simeq \mathcal{L}L(V_\tau)$ and in particular, $\mathcal{R}_{\sigma, \tau} = \mathcal{S}_{\sigma, \tau}$.

Now, appealing to Lemma 6, we conclude that
$$\mathcal{S} = \mathcal{R} = \prod \mathcal{S}U(X_\lambda)$$ where $\lambda$ ranges over a certain subset of the homomorphisms $E^+ \otimes \mathbb{R} \rightarrow \mathbb{R}$. Since $S$ and $R$ are connected, and $S \subseteq R$, it follows that $S = R$. This proves the Theorem.

Remark. In fact, it can be shown that $H_\sigma = GL(V_\sigma)$ and that the dimension of the center of $H$ is at least 2. Hence, if $\dim \mathbb{Q} E \leq 4$, then $\text{Hod}(A) = L(A)$.

The next theorem is a slight generalization of a result of Tanke'ev [11] who proved it in the case $\Delta(A) = E$. The case $m = 1$ is a special case of [5, Theorem 4.1].
THEOREM 2. Suppose that $E$ is a product of totally real fields and that $V$ is free over $E$ of rank $2m$, $m$ odd. Then $\text{Hod}(A) = L(A)$. In particular, $\mathcal{H}(A^k) = \mathcal{D}(A^k)$ for all $k \geq 1$.

PROOF. As before, for each homomorphism $\sigma: E \otimes \mathbb{C} \to \mathbb{C}$, we have the $H$ module $V_\sigma$ which is a $2m$ dimensional $\mathbb{C}$ vector space. Write $H_\sigma \sim H_{1,\sigma} \times \cdots \times H_{r,\sigma}$ where each $H_{i,\sigma}$ is a simple group. Write $V_\sigma = V_{1,\sigma} \otimes \cdots \otimes V_{r,\sigma}$ for the corresponding decomposition of $V_\sigma$. By Lemma 1, each $V_\sigma$ is a simple $H_\sigma$ module. As $E$ is a product of totally real fields, the polarization $\psi$ induces an alternating form $\psi_\sigma$ on $V_\sigma$ and $H_\sigma \subseteq L_\sigma \subseteq \text{Sp}(V_\sigma, \psi_\sigma)$. (Here, $L_\sigma$ is the projection of $L$ to $\text{GL}(V_\sigma)$.) Hence, each $V_{i,\sigma}$ is either a symplectic or orthogonal representation of $H_{i,\sigma}$. Since a symplectic representation has even dimension, exactly one of the $V_{i,\sigma}$ is symplectic and the others must be orthogonal. From [10, Proposition 7 and the Appendix], we see that when $m$ is odd, there are the following possibilities:

(i) $H_{i,\sigma}$ is of type $A_n$ ($n \geq 1$) and $V_{i,\sigma}$ is an exterior power of the standard representation. Here, $\dim V_{i,\sigma} = \binom{n+1}{k}$ for some $k \leq n$.

(ii) $H_{i,\sigma}$ is of type $C_n$ ($n \geq 2$) and $V_{i,\sigma}$ is the standard representation. Here, $\dim V_{i,\sigma} = 2n$ and $V_{i,\sigma}$ is symplectic.

(iii) $H_{i,\sigma}$ is of type $D_n$ ($n \geq 4$) and $V_{i,\sigma}$ is the standard representation. Here, $\dim V_{i,\sigma} = 2n$ and $V_{i,\sigma}$ is orthogonal.

Suppose $H_{i,\sigma}$ is of type $A_n$. For $V_{i,\sigma}$ to be symplectic or orthogonal, we must have $n$ odd, say $n + 1 = 2n_0$, and $k = n_0$. Moreover, $V_{i,\sigma}$ is symplectic when $n_0$ is odd and orthogonal when $n_0$ is even. Now,

$$
\dim V_{i,\sigma} = \binom{n+1}{k} = \binom{2n_0}{n_0}
$$

is always even and is divisible by 4 if $n_0$ is odd. Thus, $m$ odd implies that $V_{i,\sigma}$ cannot be symplectic. But in this case, some other $V_{j,\sigma}$ is symplectic and this would give $4 \mid 2m$ and so, case (i) cannot occur. But, as $\dim V_{i,\sigma}$ is even in both the remaining cases, we must have $r = 1$ and $H_\sigma$ of type $C_n$, i.e. $H_\sigma = \text{Sp}(V_\sigma, \psi_\sigma) = L_\sigma$. Again, by taking Lie algebras and appealing to Lemmas 1, 5 and 6, it follows that

$$
H = L = \prod \text{Sp}(V_\sigma, \psi_\sigma).
$$

Finally, we check that no simple factor of $A$ is of type III by a dimension calculation as in the proof of Theorem 1. Now by [6, Theorem 3.1] (stated as (**) in the Introduction), it follows that $\mathcal{H}(A^k) = \mathcal{D}(A^k)$ for all $k \geq 1$. This proves the theorem.

Finally, we combine the two theorems to prove the result stated in the Introduction.

PROOF OF THE PROPOSITION. As $A$ is simple, any maximal commutative semisimple subalgebra of $\Delta(A)$ must in fact be a field $E$, (say). Moreover, $E$ is totally real or a CM field. In the first case, $E$ acts on $\text{Lie}(A)$ and so $[E : \mathbb{Q}]$ divides $\dim A$. Thus, $\dim_E V = 2m$, $m$ odd. The Proposition follows in this case from Theorem 2. In the second case, $[E : \mathbb{Q}]$ is even and so $\dim_E V = m$ is odd. The Proposition follows in this case from Theorem 1.
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