GENERALIZED OPEN MAPPING THEOREMS FOR BILINEAR MAPS, WITH AN APPLICATION TO OPERATOR ALGEBRAS

P. G. DIXON

(Communicated by John B. Conway)

ABSTRACT. Cohen [4] gave an example of a surjective bilinear mapping between Banach spaces which was not open, and Horowitz [8] gave a much simpler example. We build on Horowitz' example to produce a similar result for bilinear mappings such that every element of the target space is a linear combination of \( n \) elements of the range. An immediate application is that Bercovici's construction [1] of an operator algebra with property \((A_1)\) but not \((A_{1/n}(r))\) can be extended to achieve property \((A_{1/n})\) without \((A_{1/n}(r))\).

Let \( T : X \times Y \rightarrow Z \) be a continuous bilinear mapping between Banach spaces. We shall be concerned with the following two properties of such a mapping.

DEFINITION. For \( n = 1, 2, 3, \ldots \) we say \( T \) is \((1/n)\)-surjective if for every \( z \in Z \) there exist \( x_1, x_2, \ldots, x_n \in X \) and \( y_1, y_2, \ldots, y_n \in Y \) with

\[
  z = T(x_1, y_1) + T(x_2, y_2) + \cdots + T(x_n, y_n).
\]

If, further, there exists a constant \( K > 0 \) such that for every \( z \) the \( x_1, \ldots, x_n \) and \( y_1, \ldots, y_n \) may be found, satisfying (1), with

\[
  \sum_{i=1}^{n} \|x_i\|\|y_i\| \leq K\|z\|,
\]

then we say that \( T \) is \((1/n)\)-open.

REMARKS. (a) These properties are generalizations of the properties \((A_{1/n})\) and \("(A_{1/n}(r))\) for some \( r > 0 \)" of operator algebras studied by Bercovici, Foiaş and Pearcy [2]; hence the "\(1/n\)" in our notation.

(b) If \( n = 1 \), "\((1/n)\)-surjective" is "surjective" and "\((1/n)\)-open" is "open".

(c) Notice that

\[
  \sum_{i=1}^{n} T(x_i, y_i) = \sum_{i=1}^{n} T(\lambda_i x_i, \lambda_i^{-1} y_i)
\]

for any nonzero scalars \( \lambda_1, \ldots, \lambda_n \). We may therefore arrange that \( \|x_i\| = \|y_i\| \) (1 \( \leq i \leq n \)). Thus, \( T \) is \((1/n)\)-open if and only if there is a constant \( L > 0 \) such that every \( z \in Z \) with \( \|z\| \leq 1 \) may be expressed in the form (1) with \( \|x_i\| \leq L, \|y_i\| \leq L \) (1 \( \leq i \leq n \)).

(d) We may work over the real field or the complex field and, in the latter case, we may study either bilinear or sesquilinear maps. The definitions above and Theorems 1 and 2 below apply to all three cases.

Received by the editors September 1, 1987.

1980 Mathematics Subject Classification (1985 Revision). Primary 46A30; Secondary 47D35.

Key words and phrases. Open mapping theorem, bilinear map, dual algebra.

©1988 American Mathematical Society

0002-9939/88 $1.00 + $0.25 per page

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
The best positive result known is the following, which is present, at least implicitly, in [9, Theorem 1.3, 7, Lemma 3.5, 3 and 5].

**Theorem 1.** Suppose \( T: X \times Y \to Z \) is a continuous bilinear (or sesquilinear) map between separable Banach spaces \( X, Y, Z \). Then

(a) if, for every \( z \in Z \), there exists a positive integer \( n \) and \( x_1, \ldots, x_n \in X, y_1, \ldots, y_n \in Y \) satisfying (1), then \( T \) is \((1/n)\)-surjective for some \( n \) (i.e. the dependence of \( n \) on \( z \) can be removed);

(b) if \( T \) is \((1/n)\)-surjective, then \( T \) is \((1/2n)\)-open.

It is known [6] that the separability condition cannot be removed from (a). We conjecture that it cannot be removed from (b) either. Horowitz [8] showed that, even for finite-dimensional \( X, Y, Z \), (b) cannot be improved to "surjective implies open" in the \( n = 1 \) case. Our main result here is the extension of his example to show that, generally, \((1/n)\)-surjective does not imply \((1/n)\)-open. It remains an interesting open question whether \((1/n)\)-surjective implies \((1/m)\)-open for \( n < m < 2n \).

**Theorem 2.** For each positive integer \( n \), there exists a \((1/n)\)-surjective sesquilinear (or bilinear) map \( T: C^{n+2} \times C^{n+2} \to C^n \) (where \( N = n^2 + n + 2 \)), which is not \((1/n)\)-open.

**Proof.** We show the existence of a sesquilinear map \( T \): a complex bilinear map \( \tilde{T} \) with the same properties is then obtained by:

\[ \tilde{T}(x, y) := T(x, \bar{y}) \quad (x, y \in C^{n+2}), \]

and a real bilinear map is obtained just by replacing \( C \) by \( R \).

We observe that a sesquilinear map \( T: X \times Y \to Z \) is \((1/n)\)-surjective (respectively, \((1/n)\)-open), if and only if the map \( T^{(n)}: X^n \times Y^n \to Z \) is surjective (respectively, open), where

\[ T^{(n)}((x_i)_{i=1}^n, (y_i)_{i=1}^n) := \sum_{i=1}^n T(x_i, y_i). \]

Our map \( T: C^{n+2} \times C^{n+2} \to C^n \) is defined by

\[
T((u_1, \ldots, u_n, v, w), (x_1, \ldots, x_n, y, z)) \\
= ((u_i \bar{x}_j)_{i,j=1}^n, (u_i \bar{y})_{i=1}^n, u_1 z + v \bar{x}_1 + w \bar{y}, w \bar{x}_1 + v \bar{y}).
\]

The reader may verify that in the case \( n = 1 \) this reduces to Horowitz' example, except for the complex conjugates and order of components. To construct \( T^{(n)} \) we simply replace the numbers \( u_1, \ldots, z \in C \) by vectors \( u_1, \ldots, z \in C^n \) and the products \( u_i \bar{x}_j \), etc., by scalar products \( u_i \cdot x_j \), etc.:  

\[
T^{(n)}((u_1, \ldots, u_n, v, w), (x_1, \ldots, x_n, y, z)) \\
= ((u_i \cdot x_j)_{i,j=1}^n, (u_i \cdot y)_{i=1}^n, u_1 \cdot z + v \cdot x_1 + w \cdot y, w \cdot x_1 + v \cdot y).
\]

To show that \( T^{(n)} \) is surjective, take any vector

\[ h = ((\xi_i)_{i,j=1}^n, (\eta_i)_{i=1}^n, \varsigma, \theta) \in C^N. \]

If \( \eta_i = 0 \) for all \( i \), then let

\[ x_j = e_j \quad (1 \leq j \leq n), \]
where \((e_j)_i = 1\) if \(i = j\), 0 if \(i \neq j\),
\[
\begin{align*}
y &= 0, \\
z &= 0, \\
u_i &= (\xi_{i1}, \xi_{i2}, \ldots, \xi_{in}) \quad (1 \leq i \leq n), \\
v &= (\zeta, 0, 0, \ldots, 0), \\
w &= (\theta, 0, 0, \ldots, 0); \\
\end{align*}
\]
otherwise, let
\[
\begin{align*}
u_i &= e_i \quad (1 \leq i \leq n) \\
x_j &= (\xi_{j1}, \xi_{j2}, \ldots, \xi_{jn}) \quad (1 \leq j \leq n), \\
y &= (\eta_1, \eta_2, \ldots, \eta_n), \\
v &= \theta(y \cdot y)^{-1}y, \\
w &= 0, \\
z &= (\zeta - (v \cdot x_1))^{-1}u_1. \\
\end{align*}
\]
In both cases, \(T^{(n)}((u_1, \ldots, w), (x_1, \ldots, z)) = h\). This proves that \(T^{(n)}\) is surjective, so \(T\) is \((1/n)\)-surjective.

To show that \(T\) is not \((1/n)\)-open, suppose otherwise. Then, by Remark (c) and the equivalence of norms on finite-dimensional spaces, there exists a number \(M > 1\) such that, whenever \(h = ((\xi_{ij}), (\eta_i), (\xi, \theta)) \in \mathbb{C}^N\) with \(|\xi_{ij}|, |\eta_i|, |\xi|, |\theta| \leq 1\) for all \(i, j\), there exist \(u_i, v, w, x_i, y, z \in \mathbb{C}^n\) \((1 \leq i \leq n)\) such that
\[
(2) \quad T^{(n)}((u_1, \ldots, u_n, v, w), (x_1, \ldots, x_n, y, z)) = h
\]
and
\[
(3) \quad \|u_i\|, \|v\|, \|w\|, \|x_i\|, \|y\|, \|z\| \leq M \quad (1 \leq i \leq n),
\]
where \(\|\cdot\|\) denotes the Euclidean norm: \(\|x\| := (x \cdot x)^{1/2}\) \((x \in \mathbb{C}^n)\).

Extending Horowitz' argument, we let \(h\) be defined by:
\[
\begin{align*}
\xi_{11} &= a, \\
\xi_{ii} &= 1 \quad (2 \leq i \leq n), \\
\xi_{ij} &= 0 \quad (i \neq j), \\
\eta_1 &= a, \\
\eta_i &= 0 \quad (2 \leq i \leq n), \\
\zeta &= a, \\
\theta &= 1; \\
\end{align*}
\]
where \(0 < a < 1\) and we shall eventually make \(a\) tend to zero. Suppose \(u_i, v, w, x_i, y, z\) satisfy (2) and (3) above. We wish to show that \(\|x_1\|\|u_1\|\) must be small. This is most conveniently done by the following matrix argument.

Define the \(n \times n\) matrices \(A = (a_1, a_2, \ldots, a_n)\), \(B = (b_1, b_2, \ldots, b_n)\) from the column vectors
\[
\begin{align*}
a_1 &= a^{-1/2}u_1, \\
a_i &= u_i \quad (2 \leq i \leq n), \\
b_1 &= a^{-1/2}x_1, \\
b_j &= x_j \quad (2 \leq j \leq n). \\
\end{align*}
\]
Then the equations \( u_i \cdot x_j = \xi_{ij} \) \((1 \leq i, j \leq n)\) become \( A^*B = I \). It follows that \( A \) and \( B \) are invertible and \( BA^* = I = B^*A \), so \( B^*BA^* = I \). Hence

\[
1 = (B^*BA^*A)_{11}
= \sum_{i=1}^{n} (b_i \cdot b_1)(a_1 \cdot a_i)
= a^{-2}||x_1||^2||u_1||^2 + \sum_{i=2}^{n} a^{-1}(x_i \cdot x_1)(u_i \cdot u_i).
\]

Since \( ||x_i||, ||u_i|| \leq M \) \((1 \leq i \leq n)\), this yields

\[
a^{-2}||x_1||^2||u_1||^2 \leq 1 + (n - 1)M^2.
\]

Hence

\[
(4) \quad ||x_1|| ||u_1|| \leq a(1 + (n - 1)M^2) \leq anM^2.
\]

Next, since \( A \) is invertible, \((u_1, u_2, \ldots, u_n)\) is a basis for \( \mathbb{C}^n \). Now \( u_i \cdot (x_1 - y) = 0 \) for all \( i \), so it follows that \( x_1 = y \). Then the equation \( w \cdot x_1 + v \cdot y = \theta = 1 \) gives

\[
(5) \quad (w + v) \cdot x_1 = 1,
\]

and so

\[
(6) \quad ||w + v|| ||x_1|| \geq 1.
\]

We also have

\[
u_1 \cdot z + v \cdot x_1 + w \cdot y = \zeta = a,
\]

so

\[
u_1 \cdot z = a - (v + w) \cdot x_1 = -(1 - a),
\]

by (5), and so

\[
(7) \quad ||u_1|| \geq (1 - a)/M.
\]

Combining (4) and (7),

\[
(8) \quad ||x_1|| \leq anM^3/(1 - a).
\]

Finally,

\[
2M \geq ||w + v||
\geq ||x_1||^{-1}, \quad \text{by (6),}
\geq (1 - a)/anM^3, \quad \text{by (8).}
\]

Now \( M \) is a constant, independent of \( a \), so letting \( a \) tend to zero gives the required contradiction, completing the proof of Theorem 2.

**Corollary.** For every positive integer \( n \), there is a commutative \((n^2 + n + 3)\)-dimensional algebra of operators on the Hilbert space \( \mathbb{C}^{2n+5} \) which has property \((A_{1/n})\), but which, for all \( r > 0 \), does not have property \((A_{1/n}(r))\).

The proof of this is the same as that of [1], where Bercovici derives this result for \( n = 1 \) from Horowitz' example. We therefore omit further discussion and refer the reader to [1] for an explanation of the terminology as well as for the proof.
REFERENCES


DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF SHEFFIELD, SHEFFIELD, S3 7RH, ENGLAND