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A FACTORIZATION THEOREM FOR UNFOLDINGS OF ANALYTIC FUNCTIONS

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ABSTRACT. Let $\tilde{f}$ and g be holomorphic function germs at 0 in $\mathbf{C}^n \times \mathbf{C}^l = \{(x,s)\}$. If $d_x g \wedge d_x \tilde{f} = 0$ and if $f(x) = \tilde{f}(x,0)$ is not a power or a unit, then there exists a germ λ at 0 in $\mathbf{C} \times \mathbf{C}^l$ such that $g(x,s) = \lambda(\tilde{f}(x,s),s)$. The result has the implication that the notion of an RL-morphism in the unfolding theory of foliation germs generalizes that of a right-left morphism in the function germ case.

The notion of an RL-morphism in the unfolding theory of foliation singularities was introduced in [**5**] to describe the determinacy results and in [**6**] the versality theorem for these morphisms is proved. This note, which should be considered as an appendix to [**5 or 6**], contains a factorization theorem implying that an RL-morphism is a generalization of a right-left morphism in the unfolding theory of function germs. It depends on the Mattei-Moussu factorization theorem [**1**] and is a generalization of a result of Moussu [**2**].

A codim 1 foliation germ at 0 in $\mathbf{C}^n$ is a module $F = (\omega)$ over the ring of holomorphic function germs generated by a germ of an integrable 1-form ω (see §2). An unfolding of F with parameter space $\mathbf{C}^m = \{t\}$ is a codim 1 foliation germ $\mathscr{F} = (\tilde{\omega})$ at 0 in $\mathbf{C}^n \times \mathbf{C}^m$ with a generator $\tilde{\omega}$ whose restriction to $\mathbf{C}^n \times \{0\}$ is ω. We let F_t be the foliation germ generated by the restriction ω_t of $\tilde{\omega}$ to $\mathbf{C}^n \times \{t\}$. Let $\mathscr{F}'$ be another unfolding of F with parameter space $\mathbf{C}^l = \{s\}$. A morphism from $\mathscr{F}'$ to $\mathscr{F}$ is a holomorphic map germ $\Phi\colon (\mathbf{C}^n \times \mathbf{C}^l, 0) \to (\mathbf{C}^n \times \mathbf{C}^m, 0)$ such that (a) $\Phi(x,s) = (\phi(s,x), \psi(s))$ for some holomorphic map germs $\phi\colon (\mathbf{C}^n \times \mathbf{C}^l, 0) \to (\mathbf{C}^n, 0)$ and $\psi\colon (\mathbf{C}^l, 0) \to (\mathbf{C}^m, 0)$, (b) $\phi(x,0) = x$ and (c) the pull back $\Phi^*\tilde{\omega}$ of $\tilde{\omega}$ by Φ generates $\mathscr{F}'$. Thus, if we set $\phi_s(x) = \phi(x,s)$, we may think of (ϕ_s) as a family of local coordinate changes of $(\mathbf{C}^n, 0)$. For an RL-morphism, in place of (c), we only require that $\phi_s^*\omega_{\psi(s)}$ generates F'_s for each s (see (2.1) Definition). Our previous result shows that if F has a generator of the form df for some holomorphic function germ f (strong first integral for F), then every unfolding of F admits a generator of the form $d\tilde{f}$ with $\tilde{f}$ an unfolding of f. In the unfolding theory of function germs, there are notions of a right morphism and a right-left morphism. The former involves coordinate changes in the source space $(\mathbf{C}^n, 0)$, whereas the latter involves coordinate changes in the target space $\mathbf{C}$ as well. It is not difficult to see that our morphism generalizes a right morphism in the sense that when F admits a strong first integral f, then it becomes a (strict) right morphism in the unfolding theory of f. For a foliation without first integrals, it may not seem relevant to talk about

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right-left morphisms. However, as stated above, our factorization theorem shows that an RL-morphism is a natural generalization of a right-left morphism, since when $F = (df)$, an RL-morphism is exactly a right-left morphism in the unfolding theory of f. We also note that RL-morphisms are closely related to integrating factors of the foliation ((2.2) Remark 2).

1. The factorization theorem. We denote by $\mathscr{O}_n$ the ring of germs of holomorphic functions at the origin 0 in $\mathbf{C}^n = \{(x_1, \ldots, x_n)\}$. A germ f in $\mathscr{O}_n$ is said to be a power if $f = f_0^m$ for some positive integer m and a nonunit f_0 in $\mathscr{O}_n$. If we denote the critical set of f by $C(f)$, then codim $C(f) \geq 2$ implies that f is not a power. We quote the following factorization theorem of Mattei and Moussu.

(1.1) THEOREM [**1**]. *Let f be a germ in $\mathscr{O}_n$ which is not a power or a unit. If g is a germ in $\mathscr{O}_n$ with $dg \wedge df = 0$, then there exists a germ λ in $\mathscr{O}_1$ such that $g = \lambda \circ f$.*

The theorem is proved using the reduction theory of singularities of holomorphic 1-forms due to Seidenberg and Van den Essen. The proof is rather simple if we assume codim $C(f) \geq 2$ (see Moussu-Tougeron [**3**]). If $\tilde{f}$ is a germ in $\mathscr{O}_{n+l}$, we may think of $\tilde{f}$ as an unfolding of $f(x) = \tilde{f}(x, 0)$ with parameter space $C^l = \{(s_1, \ldots, s_l)\}$. We denote by d_x the exterior derivation with respect to x; $d_x\tilde{f} = \sum_{i=1}^n \partial\tilde{f}/\partial x_i(s, x)dx_i$.

(1.2) THEOREM. *Let $\tilde{f}$ be a germ in $\mathscr{O}_{n+l}$ such that $f(x) = \tilde{f}(x, 0)$ is not a power or a unit in $\mathscr{O}_n$. If g is a germ in $\mathscr{O}_{n+l}$ with $d_xg \wedge d_x\tilde{f} = 0$, then there exists a germ λ in $\mathscr{O}_{1+l}$ such that $g(x, s) = \lambda(\tilde{f}(x, s), s)$.*

PROOF. First we show the existence of λ as a formal power series in s. Thus we express λ as

$$\lambda(y, s) = \sum_{|\nu| \geq 0} \lambda^{(\nu)}(y)s^\nu, \qquad \lambda^{(\nu)} \in \mathscr{O}_1,$$

where ν denotes an l-tuple $(\nu_1, \ldots, \nu_l)$ of nonnegative integers, $|\nu| = \nu_1 + \cdots + \nu_l$ and $s^\nu = s_1^{\nu_1} \cdots s_l^{\nu_l}$. In general, if $\sigma = \sum_{|\nu| \geq 0} \sigma^{(\nu)}s^\nu$ is a series in s with $\sigma^{(\nu)} \in \mathscr{O}_n^r$ for some r, we set

$$[\sigma]_p = \sum_{|\nu|=p} \sigma^{(\nu)}s^\nu \quad \text{and} \quad \sigma^{|p} = \sum_{|\nu|=0}^{p} \sigma^{(\nu)}s^\nu$$

for a nonnegative integer p.

We look for λ satisfying the congruence

$$g(x, s) \equiv_p \lambda^{|p}(\tilde{f}(x, s), s) \tag{1.3$_p$}$$

for $p \geq 0$, where $\equiv_p$ denotes equality mod s^ν, $|\nu| = p + 1$. First, $(1.3)_0$ is equivalent to

$$g(x, 0) = \lambda^{(0)}(f(x)).$$

From the condition of the theorem, we have $d(g(x, 0)) \wedge df = 0$. Hence by (1.1), there exists a germ $\lambda^{(0)}$ in $\mathscr{O}_1$ satisfying the above. Now we suppose that we have $\lambda^{|p}$ satisfying $(1.3)_p$ and look for $[\lambda]_{p+1}$. The congruence $(1.3)_{p+1}$ reads

$$g(x, s) \equiv_{p+1} \sum_{|\nu|=p+1} \lambda^{(\nu)}(f(x))s^\nu + \lambda^{|p}(\tilde{f}(x, s), s).$$

Hence, for our purpose, it suffices to show that

$$d_x[g(x,s) - \lambda^{|p}(\tilde{f}(x,s),s)]_{p+1} \wedge df = 0. \tag{1.4}$$

By $(1.3)_p$, we have (1.4) if we show that

$$d_x(g(x,s) - \lambda^{|p}(\tilde{f}(x,s),s)) \wedge d_x\tilde{f} \equiv_{p+1} 0.$$

But this follows from the condition of the theorem and

$$d_x\lambda^{|p}(\tilde{f}(x,s),s) = \frac{\partial\lambda^{|p}}{\partial y}(\tilde{f}(x,s),s)d_x\tilde{f}.$$

Thus we have a formal power series

$$\lambda(y,s) = \sum_{|\nu|\geq 0} \lambda^{(\nu)}(y)s^\nu, \qquad \lambda^{(\nu)} \in \mathscr{O}_1,$$

in s such that $g(x,s) = \lambda(\tilde{f}(x,s),s)$ as power series in (x,s). Since $\tilde{f}$ and g are both convergent, λ must be also convergent.

(1.5) REMARKS. 1. The germ λ is determined uniquely by g (and $\tilde{f}$). If we assume that $g(x,0) = f(x)$, then $\lambda(y,0) = y$.

2. The above theorem generalizes Corollaire 1 in [**2**, Chapter II, 1] in the case $X = H$.

2. Some types of morphisms in the unfolding theory of foliation germs. We denote by Ω_n the $\mathscr{O}_n$-modules of germs of holomorphic 1-forms at 0 in $\mathbf{C}^n$. We recall (cf. [**4**, **5**]) that a codim 1 foliation germ at 0 in $\mathbf{C}^n$ is a rank 1 free sub-$\mathscr{O}_n$-module $F = (\omega)$ of Ω_n with a generator satisfying the integrability condition $d\omega \wedge \omega = 0$. The singular set $S(F)$ of F is defined to be the singular set $\{x \mid \omega(x) = 0\}$ of ω. We always assume that codim $S(F) \geq 2$. An unfolding of $F = (\omega)$ is a codim 1 foliation germ $\mathscr{F} = (\tilde{\omega})$ at 0 in $\mathbf{C}^n \times \mathbf{C}^m = \{(x,t)\}$, for some m, with a generator $\tilde{\omega}$ satisfying $\iota^*\tilde{\omega} = \omega$, where ι denotes the embedding of $\mathbf{C}^n$ into $\mathbf{C}^n \times \mathbf{C}^m$ given by $\iota(x) = (x,0)$. We call $\mathbf{C}^m$ the parameter space of $\mathscr{F}$. We recall the following definition [**5**, (2.1), **6**, (1.1)].

(2.1) DEFINITION. Let $\mathscr{F}$ and $\mathscr{F}'$ be two unfoldings of F with parameter spaces $\mathbf{C}^m$ and $\mathbf{C}^l = \{(s_1,\ldots,s_l)\}$, respectively.

(I) An RL-morphism from $\mathscr{F}'$ to $\mathscr{F}$ is a pair (Φ,ψ) satisfying the following conditions.

(a) Φ and ψ are holomorphic map germs making the diagram

$$\begin{array}{ccc} (\mathbf{C}^n \times \mathbf{C}^l, 0) & \xrightarrow{\Phi} & (\mathbf{C}^n \times \mathbf{C}^m, 0) \\ \downarrow & & \downarrow \\ (\mathbf{C}^l, 0) & \xrightarrow{\psi} & (\mathbf{C}^m, 0) \end{array}$$

commutative, where the vertical maps are the projections.

(b) $\Phi(x,0) = (x,0)$.

(c) For any generator $\tilde{\omega}$ of $\mathscr{F}$, there is a germ $\alpha = (\alpha_1,\ldots,\alpha_l)$ in $\mathscr{O}^l_{n+l}$ such that the germ

$$\Phi^*\tilde{\omega} + \sum_{k=1}^{l} \alpha_k ds_k$$

generates $\mathscr{F}'$.

(II) A morphism from $\mathscr{F}'$ to $\mathscr{F}$ is an RL-morphism such that for any generator $\tilde{\omega}$ of $\mathscr{F}$, we may choose $\alpha = 0$ in (c).

(2.2) REMARKS. 1. In both cases, we may replace "any" by "some".

2. From the integrability condition we see that, for α in (c), each $\alpha_k(x,0)$ is an integrating factor of $\omega = \iota^*\tilde{\omega}$, i.e., $\alpha_k(x,0)d\omega = d(\alpha_k(x,0)) \wedge \omega$.

3. We have a "versality theorem" for each type of morphisms [4, 6].

If a germ $\tilde{f}$ in $\mathscr{O}_{n+m}$ is an unfolding of f, i.e., if $\iota^*\tilde{f} = f$, then $\mathscr{F} = (d\tilde{f})$ is an unfolding of $F = (df)$ with parameter space $\mathbf{C}^m$ and conversely, any unfolding of $F = (df)$ has a generator of the form $d\tilde{f}$ with $\tilde{f}$ an unfolding of f [4, p. 47]. We recall the following definition (cf. [7, Definition 3.2]).

(2.3) DEFINITION. Let $\tilde{f}$ and g be two unfoldings of f with parameter spaces $\mathbf{C}^m$ and $\mathbf{C}^l = \{(s_1, \dots, s_l)\}$, respectively.

(I) A right-left morphism from g to $\tilde{f}$ is a pair (Φ, ψ) satisfying (I)(a) and (b) in (2.1) Definition and

(c) $g(x,s) = \lambda(\Phi^*\tilde{f}(x,s), s)$ for some λ in $\mathscr{O}_{1+l}$ with $\lambda(y,0) = y$.

(II) A strict right morphism from g to $\tilde{f}$ is a right-left morphism such that $\lambda(y,s) = y$ in (c).

The following is a direct consequence of (1.2) Theorem.

(2.4) PROPOSITION. *Let $\tilde{f}$ and g be unfoldings of f. A pair (Φ, ψ) is, respectively, a right-left morphism or a strict right morphism from g to $\tilde{f}$ if and only if it is an RL-morphism or a morphism from $\mathscr{F}' = (dg)$ to $\mathscr{F} = (d\tilde{f})$.*

REFERENCES

1. J.-F. Mattei et R. Moussu, *Holonomie et intégrales premières*, Ann. Sci. Ecole Norm. Sup. **13** (1980), 469–523.
2. R. Moussu, *Sur l'existence d'intégrales premières pour un germe de forme de Pfaff*, Ann. Inst. Fourier (Grenoble) **26** (1976), 171–220.
3. R. Moussu et J.-C. Tougeron, *Fonctions composées analytiques et defférentiables*, C. R. Acad. Sci. Paris **282** (1976), 1237–1240.
4. T. Suwa, *A theorem of versality for unfoldings of complex analytic foliation singularities*, Invent. Math. **65** (1981), 29–48.
5. ——, *Determinacy of analytic foliation germs*, Advanced Studies in Pure Math., vol. 5, Foliations (I. Tamura, ed.), Kinokuniya and North-Holland, 1985, pp. 427–460.
6. ——, *The versality theorem for RL-morphisms of foliation unfoldings*, Advanced Studies in Pure Math., vol. 8, Complex Analytic Singularities (T. Suwa and P. Wagreich, eds.), Kinokuniya and North-Holland, 1986, pp. 599–631.
7. G. Wassermann, *Stability of unfoldings*, Lecture Notes in Math., vol. 393, Springer-Verlag, Berlin and New York, 1974.

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