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COMPLEX RETRACTIONS AND ENVELOPES OF HOLOMORPHY

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ABSTRACT. In this paper we show that if a domain Ω of a Stein manifold X is a "holomorphic deformation retract" of a domain of holomorphy $D \subseteq X$, then Ω has a univalent envelope of holomorphy $\Omega^* \subseteq X$.

1. Introduction. Let X be a connected Stein manifold of dimension $n > 1$. Let D and Ω be domains in X with $\Omega \subseteq D$. In this paper we study the pairs $(D; \Omega)$ such that D is a domain of holomorphy which can be deformed inside Ω with a continuous family of holomorphic maps (see Definition 1). It will be shown that the domain Ω has a univalent envelope of holomorphy $\Omega^* \subseteq X$. Under some additional hypotheses, we shall prove that $(D; \Omega^*)$ is a Runge pair.

2.

DEFINITION 1. Let $\Omega \subseteq D$ be domains of a Stein manifold X; we say that the continuous map $F\colon [0;1] \times D \to X$ is a complex retraction of D in Ω if the following conditions are fulfilled:
(1) $F(0) = \mathrm{Id}$,
(2) $F(1; D) \subseteq \Omega$,
(3) $F(t)\colon D \to X$ is holomorphic, $\forall t \in [0;1]$,
(4) $F(t; \Omega) \subseteq \Omega$, $\forall t \in [0;1]$.

THEOREM 1. *Let $D \supseteq \Omega$ be a pair of domains in X with D being a domain of holomorphy; let $F\colon [0;1] \times D \to X$ be a complex retraction of D in Ω. Then Ω has a univalent envelope of holomorphy $\Omega^* \subseteq D \subseteq X$ and $F(t; \Omega^*) \subseteq \Omega^*$, $\forall t \in [0;1]$.*

LEMMA 1. *Let V and W be Stein manifolds, let $\Delta_1 \subseteq V$ and $\Delta_2 \subseteq W$ be domains, let $(\tilde{\Delta}_1; \pi_1)$ and $(\tilde{\Delta}_2; \pi_2)$ their Riemann domains envelopes of holomorphy, let $i_1\colon \Delta_1 \to \tilde{\Delta}_1$, $i_2\colon \Delta_2 \to \tilde{\Delta}_2$ be the canonical embeddings.*

If $f\colon V \to W$ is a holomorphic map such that $f(\Delta_1) \subseteq \Delta_2$, then there exists a unique map $\tilde{f}\colon \tilde{\Delta}_1 \to \tilde{\Delta}_2$ such that the diagram

$$\begin{array}{ccc}
\Delta_1 & \xrightarrow{f_{|\Delta_1}} & \Delta_2 \\
{\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i_2} \\
\tilde{\Delta}_1 & \xrightarrow{\tilde{f}} & \tilde{\Delta}_2 \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\
V & \xrightarrow{f} & W
\end{array}$$

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is commutative. So, $f(\pi_1(\tilde{\Delta}_1)) \subseteq \pi_2(\tilde{\Delta}_2)$. *Moreover, if* $F\colon [0;1] \times V \to W$ *is continuous,* $F(t)$ *is holomorphic* $\forall t \in [0;1]$ *and* $F(t)(\Delta_1) \subseteq \Delta_2$ $\forall t \in [0;1]$, *then* $\tilde{F}\colon [0;1] \times \tilde{\Delta}_1 \to \tilde{\Delta}_2$ *is continuous.*

PROOF. Let us take the map $\tilde{f} = i_2 \circ f \circ \pi_1 \colon i_1(\Delta_1) \to \tilde{\Delta}_2$. Now regard $\tilde{\Delta}_2$ as a closed submanifold of $\mathbf{C}^N$ $(N = 2n+1)$ and consider $\tilde{f}\colon i_1(\Delta_1) \to \mathbf{C}^N$. This map can be extended to $\tilde{f}\colon \tilde{\Delta}_1 \to \mathbf{C}^N$. We must prove that $\tilde{f}(\tilde{\Delta}_1) \subseteq \tilde{\Delta}_2$. Choose a point $p \in \mathbf{C}^N - \tilde{\Delta}_2$ and an open set of holomorphy $A \subseteq \mathbf{C}^N$ such that $p \notin A \supseteq \tilde{\Delta}_2$; we know that $\tilde{f}^{-1}(A)$ is a Stein open set in $\tilde{\Delta}_1$ containing $i_1(\Delta_1)$, so $\tilde{f}^{-1}(A) = \tilde{\Delta}_1$ and $p \notin \tilde{f}(\tilde{\Delta}_1)$.

Commutativity of the diagram and uniqueness of $\tilde{f}$ follow from uniqueness of analytic continuation. Now let $F\colon [0;1] \times V \to W$ be as above, and take $t_n \to t_0$ in $[0;1]$. It follows from Vitali's theorem that $F(t_n) \to F(t_0)$ uniformly on the compact sets of V, and thus $\tilde{F}(t_n) \to \tilde{F}(t_0)$ on the compact sets of $i_1(\Delta_1)$.

Let $\{K_n\}_{n\in N}$ be an exhaustion of $i_1(\Delta_1)$ by compact sets, then $\bigcup_{n\in N} \overset{\circ}{(\hat{K}_n)_{\tilde{\Delta}_1}}$ is an open subset of holomorphy of $\tilde{\Delta}_1$. It follows that $\tilde{F}(t_n) \to \tilde{F}(t_0)$ on the compact sets of $\tilde{\Delta}_1$ and, thus, $\tilde{F}\colon [0;1] \times \tilde{\Delta}_1 \to \tilde{\Delta}_2$ is continuous.

PROOF OF THEOREM 1. Let $(\tilde{\Omega};\pi)$ be the envelope of holomorphy of Ω; we have $\Omega^* = \pi(\tilde{\Omega}) \subseteq D$. Choose a point $0 \in \Omega$. A curve γ in X joining 0 and a point $P \in \Omega^*$ is such that there exists a curve $\tilde{\gamma}$ starting from $0 \in \tilde{\Omega}$ with $\pi(\tilde{\gamma}) = \gamma$ if and only if every holomorphic function $f\colon \Omega \to \mathbf{C}$ can be holomorphically continued along γ [**4**]. We will call this kind of curve a liftable curve. Let us consider the continuous map $\tilde{F}\colon [0;1] \times \tilde{\Omega} \to \tilde{\Omega}$ defined by Lemma 1, where $V = D$ and $W = X$. Take a liftable curve $\gamma = \pi(\tilde{\gamma})$ joining $0 \in \Omega$ and $P \in \pi(\tilde{\Omega})$. Then $F(t;\gamma) = F(t;\pi(\tilde{\gamma})) = \pi\tilde{F}(t;\tilde{\gamma})$ is a liftable curve joining $F(t;0)$ and $F(t;p)$. The curve $C_p(t) = F(1-t;\gamma(1)) = F(1-t;\pi\tilde{\gamma}(1)) = \pi(\tilde{F}(1-t;\gamma(1)))$ is also liftable. Moreover, the curve $C_0(t) = F(t;0)$ has support in Ω. We now define the curve $\phi(t;\gamma)$ in the following way:

$$\begin{aligned} \phi(t;\gamma)(s) &= C_0(3st) && \text{if } 0 \le s \le \tfrac{1}{3},\\ \phi(t;\gamma)(s) &= F(t;\gamma(3s-1)) && \text{if } \tfrac{1}{3} \le s \le \tfrac{2}{3},\\ \phi(t;\gamma)(s) &= C_p(3t(s-1)+1) && \text{if } \tfrac{2}{3} \le s \le 1. \end{aligned}$$

Every curve $\phi(t;\gamma)$ is a liftable curve joining 0 and P. The map $\phi(t;s) = \phi(t;\gamma)(s)$ is a homotopy between γ and $\phi(1;\gamma)$. Now, if we take a function f holomorphic in Ω and extend this function along $\phi(1;\gamma)$, we see that the value in P of the extension does not depend on γ, but only on C_p. Hence the monodromy theorem [**4**] implies that the value in P of the extension of f along γ does not depend on γ. This is our thesis. The fact that $F(t;\Omega^*) \subseteq \Omega^*$ $\forall t \in [0;1]$ follows by Lemma 1.

THEOREM 2. *Let* $\Delta \subseteq \Omega$ *be domains of* X *with* Δ *a domain of holomorphy, and assume that there exists a continuous map* $F\colon [0;1] \times \Omega \to \Omega$ *which fulfills conditions* 1, 2 *and* 3 *of Definition* 1 *with respect to the pair* $(\Omega;\Delta)$. *Under these assumptions, the following facts are equivalent*:

(a) *The domain* Ω *has a univalent envelope of holomorphy* $\Omega^* \subseteq X$.

(b) *There exists a domain of holomorphy* $D \supseteq \Omega$ *and a family* $\tilde{F}(t)$, *of holomorphic maps from* D *to* X, $t \in [0;1]$ *such that* $\tilde{F}(t)$ *extends* $F(t)$ *for every* $t \in [0;1]$.

(c) *There exists a domain of holomorphy* $D \supseteq \Omega$ *and a complex retraction of* D *in* Ω.

PROOF. (a)$\Rightarrow$(b) follows by Lemma 1.

(c)$\Rightarrow$(a) by Theorem 1.

(b)$\Rightarrow$(c). Consider the set L equal to the intersection of all the domains of holomorphy containing Ω; then, since Ω is open and connected, the interior of L, which we shall denote by $\tilde{D}$, is a domain of holomorphy contained in D [**7**]. The maps $\tilde{F}(t)\colon \tilde{D} \to X$ define a map $\tilde{F}\colon [0;1] \times \tilde{D} \to X$. We wish to prove that $\tilde{F}$ is a complex retraction of $\tilde{D}$ in Ω. We then have to check conditions 1 to 4 of Definition 1. Condition 1 follows from the uniqueness of analytic continuation. We know that $(\tilde{F}(1))^{-1}(\Delta)$ is an open set of holomorphy containing Ω, then $\tilde{D} \subseteq (\tilde{F}(1))^{-1}(\Delta)$ and $\tilde{F}(1;\tilde{D}) \subseteq \Delta \subseteq \Omega$. Therefore, condition 2 is verified.

Conditions (3) and (4) follow by the definition of $\tilde{F}$. Continuity of $\tilde{F}$ follows as shown in Lemma 1.

THEOREM 3. *Let* $\Omega \subseteq D$ *be domains of* X; *suppose that* D *is a domain of holomorphy; let* $F\colon [0;1] \times D \to X$ *be a complex retraction of* D *in* Ω *and assume that* F *fulfills the following additional conditions*:

(i) $F(t;D) \subseteq D\ \forall t \in [0;1]$,

(ii) $F(t)$ *is one-to-one* $\forall t \in [0;1[$,

(iii) $(D;F(t;D))$ *is a Runge pair* $\forall t \in [0;1[$.

Then, if Ω^* *denotes the envelope of* Ω, $(D;\Omega^*)$ *is a Runge pair.*

PROOF. Let $K \subseteq \Omega^*$ be a compact set; we must prove that $\hat{K}_D \subseteq \Omega^*$. By Theorem 1, we have established that $F(t;\Omega^*) \subseteq \Omega^*$, $\forall t \in [0;1]$. Observe that the set $K_1 = (F([0;1] \times K))\widehat{_{\Omega^*}}$ is compact in Ω^*. Notice now that if we choose $t \in [0;1[$ such that $F(t;\hat{K}_D) \subseteq \Omega^*$, then $F(t;\hat{K}_D) \subseteq K_1$. Indeed, since $F(t)\colon D \to F(t;D)$ is a biholomorphic map and $(D;F(t;D))$ is a Runge pair, we have

$$F(t;\hat{K}_D) = (F(t;K))\widehat{_{F(t;D)}} = F(t;K))\widehat{_D}.$$

Therefore, the inclusion $(F(t;K))\widehat{_D} \subseteq \Omega^*$ implies that $(F(t;K))\widehat{_D} \subseteq (F(t;K))\widehat{_{\Omega^*}} \subseteq K_1$ [**5**].

Now observe that the set $A = \{t \in [0;1]\colon F(t;\hat{K}_D) \subseteq \Omega^*\}$ is open and nonempty in $[0;1]$; in fact, $1 \in A$ and if $t_0 \in A$, since $\hat{K}_D$ is compact and Ω^* is open, we can find an open neighborhood of t_0 in $[0;1]$ contained in A. Hence, $A - \{1\}$ is open and nonempty in $[0;1[$; moreover $A - \{1\} = \{t \in [0;1[\colon F(t;\hat{K}_D) \subseteq K_1\}$, which is closed in $[0;1[$. Therefore $A = [0;1]$ and, thus, $F(0;\hat{K}_D) = \hat{K}_D \subseteq \Omega^*$.

EXAMPLES. (1) Let $X = \mathbf{C}^N$, let Ω be an open set containing 0, and star-shaped with respect to 0. Then Ω has a univalent envelope of holomorphy $\Omega^* \subseteq \mathbf{C}^N$ and $(\Omega^*;\mathbf{C}^N)$ is a Runge pair. This is a consequence of Theorems (1) and (3) considering the following map, $F\colon [0;1] \times \mathbf{C}^N \to \mathbf{C}^N\colon F(t;z) = (1-t)z$. A different proof of this fact was given in [**1**].

(2) Let $X = \mathbf{C}^N$, and let Ω be a domain satisfying the following properties:

(a) There exists $(\alpha_1 \cdots \alpha_n) \in \mathbf{N}^n$ such that if $z \in \Omega$ and $t \in [0;1]$, $(t^{\alpha_1}z_1; t^{\alpha_2}z_2;\ldots;t^{\alpha_n}z_n) \in \Omega$.

(b) If $(i_1 \cdots i_k)$ is the ordered set of indices such that $\alpha_{i_k} = 0$ and $\pi_{i_1 \cdots i_k} : \mathbf{C}^n \to \mathbf{C}^k$ denotes the canonical projection, then $\pi_{i_1 \cdots i_k}(\Omega)$ is a domain of holomorphy.

Under these assumptions, the domain Ω has a univalent envelope of holomorphy $\Omega^* \subseteq \mathbf{C}^n$ and $(\pi_{i_1 \cdots i_k}(\Omega) \times \mathbf{C}^{n-k}; \Omega^*)$ is a Runge pair.

This can be shown by considering the map $F\colon [0;1] \times \pi_{i_1 \cdots i_k}(\Omega) \times \mathbf{C}^{n-k} \to \mathbf{C}^n$:

$$F(t; z_1; \dots ; z_n) = ((1-t)^{\alpha_1} z_1; \dots ; (1-t)^{\alpha_n} z_n).$$

COUNTEREXAMPLE. Note that if $\Omega \subseteq \mathbf{C}^n$ is star-shaped with respect to the origin, then Ω fulfills the hypotheses of Theorem 2; therefore, every domain that is biholomorphic to a star-shaped domain fulfills the same hypotheses. We will now exhibit a domain $\Omega \subseteq \mathbf{C}^2$ which is biholomorphic to a star-shaped domain; however, it does not have a univalent envelope of holomorphy in $\mathbf{C}^2$. Let $\Omega = \{(z;w) \in \mathbf{C}^2 \colon ||z|-1| < \frac{1}{2}; \exists \theta > 0 \text{ such that } z = |z|e^{i\theta} \text{ and } ||w| - \theta| < \frac{1}{2}\}$. It is well known that Ω has no univalent envelope in $\mathbf{C}^2$ [**3**]. However, the map $F(z;w) = (\log z; w)$ is a biholomorphism of Ω with a star-shaped domain.

ADDENDUM. In a forthcoming paper E. Casadio Tarabusi gives an example of a domain $\Omega \subseteq \mathbf{C}^2$ with the following properties:

(a) Ω has a univalent envelope of holomorphy $\Omega^* \subseteq \mathbf{C}^2$;

(b) There exists a continuous map $F\colon [0,1] \times \Omega^* \to \Omega^*$ fulfilling conditions (1), (2), (4) of Definition 1 (in fact Ω is a deformation retract of Ω^*);

(c) There is no complex retraction of Ω^* in Ω.

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