COMPLEX RETRACTIONS AND ENVELOPES
OF HOLOMORPHY

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(Communicated by Irwin Kra)

ABSTRACT. In this paper we show that if a domain $\Omega$ of a Stein manifold $X$ is a “holomorphic deformation retract” of a domain of holomorphy $D \subseteq X$, then $\Omega$ has a univalent envelope of holomorphy $\Omega^* \subseteq X$.

1. Introduction. Let $X$ be a connected Stein manifold of dimension $n > 1$. Let $D$ and $\Omega$ be domains in $X$ with $\Omega \subseteq D$. In this paper we study the pairs $(D; \Omega)$ such that $D$ is a domain of holomorphy which can be deformed inside $\Omega$ with a continuous family of holomorphic maps (see Definition 1). It will be shown that the domain $\Omega$ has a univalent envelope of holomorphy $\Omega^* \subseteq X$. Under some additional hypotheses, we shall prove that $(D; \Omega^*)$ is a Runge pair.

2. DEFINITION 1. Let $\Omega \subseteq D$ be domains of a Stein manifold $X$; we say that the continuous map $F: [0; 1] \times D \rightarrow X$ is a complex retraction of $D$ in $\Omega$ if the following conditions are fulfilled:

1. $F(0) = \text{Id}$,
2. $F(1; D) \subseteq \Omega$,
3. $F(t): D \rightarrow X$ is holomorphic, $\forall t \in [0; 1]$,
4. $F(t; \Omega) \subseteq \Omega$, $\forall t \in [0; 1]$.

THEOREM 1. Let $D \supseteq \Omega$ be a pair of domains in $X$ with $D$ being a domain of holomorphy; let $F: [0; 1] \times D \rightarrow X$ be a complex retraction of $D$ in $\Omega$. Then $\Omega$ has a univalent envelope of holomorphy $\Omega^* \subseteq D \subseteq X$ and $F(t; \Omega^*) \subseteq \Omega^*$, $\forall t \in [0; 1]$.

LEMMA 1. Let $V$ and $W$ be Stein manifolds, let $\Delta_1 \subseteq V$ and $\Delta_2 \subseteq W$ be domains, let $(\Delta_1; \pi_1)$ and $(\Delta_2; \pi_2)$ their Riemann domains envelopes of holomorphy, let $i_1: \Delta_1 \rightarrow \tilde{\Delta}_1$, $i_2: \Delta_2 \rightarrow \tilde{\Delta}_2$ be the canonical embeddings.

If $f: V \rightarrow W$ is a holomorphic map such that $f(\Delta_1) \subseteq \Delta_2$, then there exists a unique map $\tilde{f}: \tilde{\Delta}_1 \rightarrow \tilde{\Delta}_2$ such that the diagram

\[
\begin{array}{c}
\Delta_1 \xrightarrow{f|_{\Delta_1}} \Delta_2 \\
i_1 \downarrow \quad \downarrow i_2 \\
\tilde{\Delta}_1 \xrightarrow{\tilde{f}} \tilde{\Delta}_2 \\
\pi_1 \downarrow \quad \downarrow \pi_2 \\
V \xrightarrow{f} W
\end{array}
\]

Received by the editors September 3, 1986 and, in revised form, July 23, 1987.


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0002-9939/88 $1.00 + \$0.25 per page

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is commutative. So, \( f(\pi_1(\hat{\Delta}_1)) \subseteq \pi_2(\hat{\Delta}_2) \). Moreover, if \( F: [0; 1] \times V \to W \) is continuous, \( F(t) \) is holomorphic \( \forall t \in [0; 1] \) and \( F(t)(\Delta_1) \subseteq \Delta_2 \ \forall t \in [0; 1] \), then \( \tilde{F}: [0; 1] \times \hat{\Delta}_1 \to \hat{\Delta}_2 \) is continuous.

PROOF. Let us take the map \( \tilde{f} = i_2 \circ f \circ \pi_1: \hat{\Delta}_1 \to \hat{\Delta}_2 \). Now regard \( \hat{\Delta}_2 \) as a closed submanifold of \( \mathbb{C}^N (N = 2n + 1) \) and consider \( \tilde{f}: i_1(\Delta_1) \to \mathbb{C}^N \). This map can be extended to \( \tilde{f}: \hat{\Delta}_1 \to \mathbb{C}^N \). We must prove that \( \tilde{f}(\Delta_1) \subseteq \hat{\Delta}_2 \). Choose a point \( p \in \mathbb{C}^N - \hat{\Delta}_2 \) and an open set of holomorphy \( A \subseteq \mathbb{C}^N \) such that \( p \notin A \supseteq \hat{\Delta}_2 \); we know that \( \tilde{f}^{-1}(A) \) is a Stein open set in \( \hat{\Delta}_1 \) containing \( i_1(\Delta_1) \), so \( \tilde{f}^{-1}(A) = \Delta_1 \) and \( p \notin \tilde{f}(\Delta_1) \).

Commutativity of the diagram and uniqueness of \( \tilde{f} \) follow from uniqueness of analytic continuation. Now let \( F: [0; 1] \times V \to W \) be as above, and take \( t_n \to t_0 \) in \( [0; 1] \). It follows from Vitali’s theorem that \( F(t_n) \to F(t_0) \) uniformly on the compact sets of \( V \), and thus \( \tilde{F}(t_n) \to \tilde{F}(t_0) \) on the compact sets of \( i_1(\Delta_1) \).

Let \( \{K_n\}_{n \in \mathbb{N}} \) be an exhaustion of \( i_1(\Delta_1) \) by compact sets, then \( \bigcup_{n \in \mathbb{N}} (K_n)_{\hat{\Delta}_1} \) is an open subset of holomorphy of \( \hat{\Delta}_1 \). It follows that \( \tilde{F}(t_n) \to \tilde{F}(t_0) \) on the compact sets of \( \hat{\Delta}_1 \) and, thus, \( \tilde{F}: [0; 1] \times \hat{\Delta}_1 \to \hat{\Delta}_2 \) is continuous.

PROOF OF THEOREM 1. Let \((\hat{\Omega}; \pi)\) be the envelope of holomorphy of \( \Omega \); we have \( \Omega^* = \pi(\hat{\Omega}) \subseteq D \). Choose a point \( 0 \in \Omega \). A curve \( \gamma \) in \( X \) joining \( 0 \) and a point \( P \in \Omega^* \) is such that there exists a curve \( \tilde{\gamma} \) starting from \( 0 \in \hat{\Omega} \) with \( \pi(\tilde{\gamma}) = \gamma \) if and only if every holomorphic function \( f: \Omega \to \mathbb{C} \) can be holomorphically continued along \( \gamma \) \([4]\). We will call this kind of curve a liftable curve. Let us consider the continuous map \( \tilde{F}: [0; 1] \times \hat{\Omega} \to \hat{\Omega} \) defined by Lemma 1, where \( V = D \) and \( W = X \). Take a liftable curve \( \gamma = \pi(\tilde{\gamma}) \) joining \( 0 \in \Omega \) and \( P \in \pi(\hat{\Omega}) \). Then \( F(t; \gamma) = F(t; \pi(\tilde{\gamma})) = \pi \tilde{F}(t; \tilde{\gamma}) \) is a liftable curve joining \( F(t; 0) \) and \( F(t; P) \). The curve \( C_p(t) = F(1-t; \gamma(1)) = F(1-t; \pi(\gamma(1))) = \pi(\tilde{F}(1-t; \gamma(1))) \) is also liftable. Moreover, the curve \( C_0(t) = F(t; 0) \) has support in \( \Omega \). We now define the curve \( \phi(t; \gamma) \) in the following way:

\[
\phi(t; \gamma)(s) = C_0(3st) \quad \text{if } 0 \leq s \leq \frac{1}{3};
\]
\[
\phi(t; \gamma)(s) = F(t; \gamma(3s - 1)) \quad \text{if } \frac{1}{3} \leq s \leq \frac{2}{3};
\]
\[
\phi(t; \gamma)(s) = C_p(3t(s - 1) + 1) \quad \text{if } \frac{2}{3} \leq s \leq 1.
\]

Every curve \( \phi(t; \gamma) \) is a liftable curve joining \( 0 \) and \( P \). The map \( \phi(t; s) = \phi(t; \gamma)(s) \) is a homotopy between \( \gamma \) and \( \phi(1; \gamma) \). Now, if we take a function \( f \) holomorphic in \( \Omega \) and extend this function along \( \phi(1; \gamma) \), we see that the value in \( P \) of the extension does not depend on \( \gamma \), but only on \( C_p \). Hence the monodromy theorem \([4]\) implies that the value in \( P \) of the extension of \( f \) along \( \gamma \) does not depend on \( \gamma \). This is our thesis. The fact that \( F(t; \Omega^*) \subseteq \Omega^* \ \forall t \in [0; 1] \) follows by Lemma 1.

THEOREM 2. Let \( \Delta \subseteq \Omega \) be domains of \( X \) with \( \Delta \) a domain of holomorphy, and assume that there exists a continuous map \( F: [0; 1] \times \Omega \to \Omega \) which fulfills conditions 1, 2 and 3 of Definition 1 with respect to the pair \( (\Omega; \Delta) \). Under these assumptions, the following facts are equivalent:

(a) The domain \( \Omega \) has a univalent envelope of holomorphy \( \Omega^* \subseteq X \).
(b) There exists a domain of holomorphy \( D \supseteq \Omega \) and a family \( \tilde{F}(t) \), of holomorphic maps from \( D \) to \( X \), \( t \in [0; 1] \) such that \( \tilde{F}(t) \) extends \( F(t) \) for every \( t \in [0; 1] \).

(c) There exists a domain of holomorphy \( D \supseteq \Omega \) and a complex retraction of \( D \) in \( \Omega \).

**Proof.** (a) \( \Rightarrow \) (b) follows by Lemma 1.

(b) \( \Rightarrow \) (c). Consider the set \( L \) equal to the intersection of all the domains of holomorphy containing \( \Omega \); then, since \( \Omega \) is open and connected, the interior of \( L \), which we shall denote by \( \hat{D} \), is a domain of holomorphy contained in \( D \). The maps \( \tilde{F}(t): \hat{D} \to X \) define a map \( F: [0; 1] \times \hat{D} \to X \). We wish to prove that \( \tilde{F} \) is a complex retraction of \( \hat{D} \) in \( \Omega \). We then have to check conditions 1 to 4 of Definition 1. Condition 1 follows from the uniqueness of analytic continuation. We know that \( (\tilde{F}(1))^{-1}(\Delta) \) is an open set of holomorphy containing \( \Omega \), then \( \hat{D} \subseteq (\tilde{F}(1))^{-1}(\Delta) \) and \( \tilde{F}(1; \hat{D}) \subseteq \Delta \subseteq \Omega \). Therefore, condition 2 is verified.

Conditions (3) and (4) follow by the definition of \( \tilde{F} \). Continuity of \( \tilde{F} \) follows as shown in Lemma 1.

**Theorem 3.** Let \( \Omega \subseteq D \) be domains of \( X \); suppose that \( D \) is a domain of holomorphy; let \( F: [0; 1] \times D \to X \) be a complex retraction of \( D \) in \( \Omega \) and assume that \( F \) fulfills the following additional conditions:

(i) \( F(t; D) \subseteq D \forall t \in [0; 1] \),

(ii) \( F(t) \) is one-to-one \( \forall t \in [0; 1] \),

(iii) \( (D; F(t; D)) \) is a Runge pair \( \forall t \in [0; 1] \).

Then, if \( \Omega^* \) denotes the envelope of \( \Omega \), \( (D; \Omega^*) \) is a Runge pair.

**Proof.** Let \( K \subseteq \Omega^* \) be a compact set; we must prove that \( \hat{K}_D \subseteq \Omega^* \). By Theorem 1, we have established that \( F(t; \Omega^*) \subseteq \Omega^*, \forall t \in [0; 1] \). Observe that the set \( K_1 = (F([0; 1] \times K))_{\hat{D}} \) is compact in \( \Omega^* \). Notice now that if we choose \( t \in [0; 1] \) such that \( F(t; \hat{K}_D) \subseteq \Omega^* \), then \( F(t; \hat{K}_D) \subseteq K_1 \). Indeed, since \( F(t): D \to F(t; D) \) is a biholomorphic map and \( (D; F(t; D)) \) is a Runge pair, we have

\[
F(t; \hat{K}_D) = (F(t; K))_{\hat{D}} = F(t; K)_{\hat{D}}.
\]

Therefore, the inclusion \( (F(t; K))_{\hat{D}} \subseteq \Omega^* \) implies that \( (F(t; K))_{\hat{D}} \subseteq (F(t; K))_{\hat{D}} \subseteq \Omega^* \).

Now observe that the set \( A = \{ t \in [0; 1] : F(t; \hat{K}_D) \subseteq \Omega^* \} \) is open and nonempty in \([0; 1]\); in fact, \( t \in A \) if \( t_0 \in A \), since \( \hat{K}_D \) is compact and \( \Omega^* \) is open, we can find an open neighborhood of \( t_0 \) in \([0; 1]\) contained in \( A \). Hence, \( A - \{ 1 \} \) is open and nonempty in \([0; 1]\); moreover \( A - \{ 1 \} = \{ t \in [0; 1] : F(t; \hat{K}_D) \subseteq K_1 \} \), which is closed in \([0; 1]\). Therefore \( A = [0; 1] \) and, thus, \( F(0; \hat{K}_D) = \hat{K}_D \subseteq \Omega^* \).

**Examples.** (1) Let \( X = \mathbb{C}^N \), let \( \Omega \) be an open set containing 0, and star-shaped with respect to 0. Then \( \Omega \) has a univalent envelope of holomorphy \( \Omega^* \subseteq \mathbb{C}^N \) and \( (\Omega^*; \mathbb{C}^N) \) is a Runge pair. This is a consequence of Theorems (1) and (3) considering the following map, \( F: [0; 1] \times \mathbb{C}^N \to \mathbb{C}^N : F(t; z) = (1 - t)z \). A different proof of this fact was given in [1].

(2) Let \( X = \mathbb{C}^N \), and let \( \Omega \) be a domain satisfying the following properties:

(a) There exists \( (\alpha_1 \cdots \alpha_n) \in \mathbb{N}^n \) such that if \( z \in \Omega \) and \( t \in [0; 1] \), \( (t^{\alpha_1} z_1; t^{\alpha_2} z_2; \ldots; t^{\alpha_n} z_n) \in \Omega \).

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(b) If \((i_1 \cdots i_k)\) is the ordered set of indices such that \(\alpha_{i_k} = 0\) and \(\pi_{i_1 \cdots i_k} : \mathbb{C}^n \to \mathbb{C}^k\) denotes the canonical projection, then \(\pi_{i_1 \cdots i_k} (\Omega)\) is a domain of holomorphy.

Under these assumptions, the domain \(\Omega\) has a univalent envelope of holomorphy \(\Omega^* \subseteq \mathbb{C}^n\), and \((\pi_{i_1 \cdots i_k} (\Omega) \times \mathbb{C}^{n-k} ; \Omega^*)\) is a Runge pair.

This can be shown by considering the map \(F : [0; 1] \times \pi_{i_1 \cdots i_k} (\Omega) \times \mathbb{C}^{n-k} \to \mathbb{C}^n:\)

\[
F(t; z_1; \ldots; z_n) = ((1-t)^{\alpha_1}z_1; \ldots; (1-t)^{\alpha_n}z_n).
\]

COUNTEREXAMPLE. Note that if \(\Omega \subseteq \mathbb{C}^n\) is star-shaped with respect to the origin, then \(\Omega\) fulfills the hypotheses of Theorem 2; therefore, every domain that is biholomorphic to a star-shaped domain fulfills the same hypotheses. We will now exhibit a domain \(\Omega \subseteq \mathbb{C}^2\) which is biholomorphic to a star-shaped domain; however, it does not have a univalent envelope of holomorphy in \(\mathbb{C}^2\). Let \(\Omega = \{(z; w) \in \mathbb{C}^2 : ||z| - 1| < \frac{1}{2}; \exists \theta > 0 \text{ such that } z = |z|e^{i\theta} \text{ and } ||w| - \theta| < \frac{1}{2}\}\).

It is well known that \(\Omega\) has no univalent envelope in \(\mathbb{C}^2\) [3]. However, the map \(F(z; w) = (\log z; w)\) is a biholomorphism of \(\Omega\) with a star-shaped domain.

ADDENDUM. In a forthcoming paper E. Casadio Tarabusi gives an example of a domain \(\Omega \subseteq \mathbb{C}^2\) with the following properties:

(a) \(\Omega\) has a univalent envelope of holomorphy \(\Omega^* \subseteq \mathbb{C}^2\);
(b) There exists a continuous map \(F : [0, 1] \times \Omega^* \to \Omega^*\) fulfilling conditions (1), (2), (4) of Definition 1 (in fact \(\Omega\) is a deformation retract of \(\Omega^*\));
(c) There is no complex retraction of \(\Omega^*\) in \(\Omega\).

BIBLIOGRAPHY


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