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L^p CONTINUITY OF PSEUDO-DIFFERENTIAL OPERATORS WITH DISCONTINUOUS SYMBOL

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ABSTRACT. A set $\Gamma \subset \mathbf{R}^n \times \mathbf{R}^n$ is called conic if $(x, \xi) \in \Gamma$ implies $(x, t\xi) \in \Gamma$, for all $t > 0$.

In localizing a partial differential equation, the following question arose: Let us consider a pseudo-differential operator L whose symbol is the characteristic function of a conic subset of $\mathbf{R}^n \times \mathbf{R}^n$. Will L be a bounded operator in $L^2(\mathbf{R}^n)$?

We present as a counterexample a pseudo-differential operator L whose symbol is the characteristic function of a conic subset of $\mathbf{R}^2 \times \mathbf{R}^2$, which is unbounded on $L^p(\mathbf{R}^2)$ for $1 \leq p \leq 2$.

Given a real-valued function c defined in an open subset D of $\mathbf{R}^2$, let us consider the following subset Γ of $\mathbf{R}^4$,

$$\Gamma = \{(x, \xi) = (x_1, x_2, \xi_1, \xi_2) | x \in D, \xi_2 > c(x)\xi_1\}.$$

Γ is a conic set. That is to say, $(x, \xi) \in \Gamma$ implies $(x, t\xi) \in \Gamma$, for all $t > 0$.

Let us now consider the pseudo-differential operator L whose symbol is the characteristic function Ψ_Γ of Γ,

$$L(f)(x) = \int e^{-2\pi i x \cdot \xi} \Psi_\Gamma(x, \xi) \hat{f}(\xi)\, d\xi, \quad \text{for } f \in C_0^\infty(\mathbf{R}^2). \tag{1}$$

We claim that for a particular selection of D and c, the operator L will be unbounded in $L^p(\mathbf{R}^2)$, for any $1 \leq p \leq 2$.

The first step in proving this claim will be to write (1) in terms of a kernel. More specifically,

LEMMA. *Let $\omega(x) = (\omega_1(x), \omega_2(x))$ be any nonzero vector in $\mathbf{R}^2$. Then, given $f \in C_0^\infty(\mathbf{R}^2)$, $x \in \mathbf{R}^2$,*

$$H(f)(x) = pv \int_{-\infty}^{\infty} f(x + t\omega(x)) \frac{dt}{t} \text{ is well defined.} \tag{2}$$

Suppose now that $\omega(x)$ is orthogonal to the line $\xi_2 = c(x)\xi_1$ and has positive components. Then,

$$L(f)(x) = \frac{i}{2\pi} H(f)(x) + \frac{1}{2} f(x). \tag{3}$$

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PROOF. If $\omega(x) \neq 0$, there exists $\varepsilon_0 = \varepsilon_0(x, \operatorname{supp}(f))$ such that

$$\int_{\varepsilon<|t|<1/\varepsilon} f(x + t\omega(x))\frac{dt}{t} = \int_{\varepsilon<|t|<\varepsilon_0} f(x + t\omega(x))\frac{dt}{t} \quad \text{for } 0 < \varepsilon < \varepsilon_0.$$

Now, if $0 < \varepsilon' < \varepsilon$,

$$\begin{aligned} &\left|\int_{\varepsilon<|t|<\varepsilon_0} f(x + t\omega(x))\frac{dt}{t} - \int_{\varepsilon'<|t|<\varepsilon_0} f(x + t\omega(x))\frac{dt}{t}\right| \\ &\qquad \leq \int_{\varepsilon'<|t|<\varepsilon} |f(x + t\omega(x)) - f(x)|\frac{dt}{t} \leq \|\nabla f\|_{L^\infty} |\omega(x)|(\varepsilon - \varepsilon'). \end{aligned}$$

This shows that (2) is pointwise well defined. To verify (3), it is enough to show that

$$H(f)(x) = \int e^{-2\pi i x\cdot\xi}(-\pi i \operatorname{sign} \omega(x) \cdot \xi)\hat{f}(\xi)\, d\xi. \tag{4}$$

Given $\varepsilon > 0$,

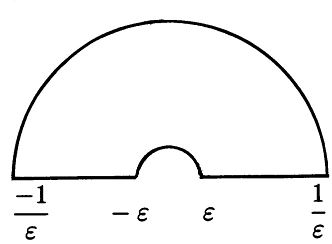
$$\begin{aligned} \int_{\varepsilon<|t|<1/\varepsilon} f(x + t\omega(x))\frac{dt}{t} &= \int_{\varepsilon<|t|<1/\varepsilon} \int e^{-2\pi i(x+t\omega(x))\cdot\xi} \hat{f}(\xi)\, d\xi\, \frac{dt}{t} \\ &= \int e^{-2\pi i x\cdot\xi} \int_{\varepsilon<|t|<1/\varepsilon} e^{-2\pi i t\omega(x)\cdot\xi}\,\frac{dt}{t}\hat{f}(\xi)\, d\xi. \end{aligned}$$

We assert that

$$\left|\int_{\varepsilon<|t|<1/\varepsilon} e^{-2\pi i t\omega(x)\cdot\xi}\frac{dt}{t}\right| \leq 2\pi. \tag{i}$$

$$\int_{\varepsilon<|t|<1/\varepsilon} e^{-2\pi i t\omega(x)\cdot\xi}\frac{dt}{t} \underset{\varepsilon\to 0}{\longrightarrow} -\pi i \operatorname{sign}(\omega(x) \cdot \xi). \tag{ii}$$

These two assertions can be derived applying the theorem of residues to the function $h(z) = (e^{-2\pi i z\omega(x)\cdot\xi})/z$ on the closed path



so we can use the dominated convergence theorem to conclude (4).

This completes the proof of the lemma.

Thus, in order to produce a counterexample, it suffices to show that for a specific choice of the set D and the function $\omega(x)$, the operator H given by (2) is unbounded in $L^p(D)$.

THEOREM. *Let D be the subset of $\mathbf{R}^2$ defined by $\{(x_1, x_2)/\delta < x_1 < x_2\}$, given $\delta > 1$. Let $\omega(x) = (1, (x_2 - 1)/(x_1 - 1))$. Then, there exists $f \in C_0^\infty(\mathbf{R}^2)$ such that $H(f) \notin L^p(D)$, $\forall 1 \le p \le 2$.*

PROOF. Let $\varphi \in C_0^\infty(\mathbf{R})$ be such that $0 \le \varphi \le 1$ and

$$\varphi(s) = \begin{cases} 1 & \text{if } 0 \le s \le 1, \\ 0 & \text{if } s \le 1 - \delta \text{ or } s \ge \delta. \end{cases}$$

Let $f(y_1, y_2) = \varphi(y_1)\varphi(y_2)$. We have to consider

$$\text{pv} \int_{-\infty}^{\infty} \varphi(x_1 + t)\varphi(x_2 + t\omega_2(x)) \frac{dt}{t}. \tag{4}$$

When $x_1 > \delta$, $\varphi(x_1 + t)$ vanishes for $t > 0$. So, (4) reduces to

$$\int_{-\infty}^{0} \varphi(x_1 + t)\varphi(x_2 + t\omega_2(x)) \frac{dt}{t}.$$

Thus,

$$|H(f)(x)| = \int_0^\infty \varphi(x_1 - s)\varphi(x_2 - s\omega_2(x)) \frac{ds}{s}.$$

Let $E = \{s > 0 | \varphi(x_1 - s)\varphi(x_2 - s\omega_2(x)) = 1\}$. Then,

$$|H(f)(x)| \ge \int_E \frac{ds}{s}$$

Given $(x_1, x_2) \in D$, E is the intersection of the intervals $(x_1 - 1, x_1)$, $(x_2 - 1/\omega_2(x), x_2/\omega_2(x))$. That is to say, E is the interval $(x_1 - 1, x_2/\omega_2(x))$. Thus,

$$|H(f)(x)| \ge \ln(x_2/(x_2 - 1)), \qquad x \in D.$$

We assert that $\ln(x_2/(x_2 - 1)) \notin L^p(D)$, for $1 \le p \le 2$. In fact,

$$\begin{aligned} &\int_\delta^\infty \int_\delta^{x_2} \ln^p \frac{x_2}{x_2 - 1} dx_1\, dx_2 \\ &\quad = \int_0^\infty \frac{1}{u} \ln^p \frac{\delta u + 1}{(\delta - 1)u + 1} \frac{du}{u^2} = \int_0^\infty \left[\frac{\ln \frac{\delta u + 1}{(\delta - 1)u + 1}}{u} \right]^p u^{p-3}\, du. \end{aligned}$$

Since the function in brackets converges to 1 as $u \to 0^+$, it will be $\ge \frac{1}{2}$ for $0 < u < \varepsilon$, for some $\varepsilon > 0$.

Thus, the integral above is $\ge \frac{1}{2} \int_0^\varepsilon u^{p-3}\, du$, which diverges when $p \le 2$.

REMARKS. (1) When the vector $\omega(x)$ does not depend on x, H is bounded in $L^p(\mathbf{R}^2)$, for every $1 < p < \infty$. (See [**1**, p. 359].)

(2) Let $p(x, \xi)$ be a bounded function. Suppose that the function $\xi \to p(x, \xi)$ is $C^\infty(\mathbf{R}^n \setminus 0)$ and homogeneous of degree 0, uniformly with respect to x. It is a classical result that the pseudo-differential operator with symbol p is bounded in $L^2(\mathbf{R}^n)$, without any hypothesis of regularity in the variable x.

However, since a subset $\Gamma \subset \mathbf{R}^n \times \mathbf{R}^n$ is conic if and only if $\Psi_\Gamma(x, \xi)$ is homogeneous of degree 0 in ξ, the above theorem shows that this result will be false without assuming some regularity in the variable ξ.

(3) When Γ is a conic subset of $\mathbf{R}^2$, the L^p continuity of the operator L, for $1 < p < \infty$, is a consequence of the Carleson-Hunt's theorem. (See [**2**].)

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REFERENCES

1. M. de Guzman, *Real variable methods in Fourier analysis*, Mathematics Studies, no. 46, North-Holland, Amsterdam, 1981.
2. O. G. Jørsboe and L. Mejlbro, *The Carleson-Hunt theorem on Fourier series*, Lecture Notes in Math., vol. 911, Springer-Verlag, Berlin and New York, 1982.

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