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AN ATOMIC DECOMPOSITION FOR PARABOLIC H^p SPACES ON PRODUCT DOMAINS

SHUICHI SATO

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ABSTRACT. We obtain an atomic decomposition for two-parameter parabolic H^p spaces, showing simultaneously an integral inequality between Lusin functions and nontangential maximal functions. As its consequence, we generalize Fefferman's weak type estimates for double singular integrals.

1. Introduction. Gundy and Stein [**6**] proved that H^p spaces on the bidisc are characterized in terms of Lusin functions. Under a certain restriction, this was extended to parabolic H^p spaces by [**7**]. In the present note we consider parabolic H^p spaces on $\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}$ again and extend R. Fefferman's atomic decomposition (see [**4**]) to this setting, which enables us to remove the restriction mentioned above and brings the Lusin function characterization of two-parameter parabolic H^p spaces. (See Theorems 1 and 2.) As a consequence, C. Fefferman's weak type estimates for double singular integrals are further generalized (see [**3**, **7**] and Theorem 3).

2. Preliminaries. Here are some of our notations and background materials.

(2.1) If $x \in \mathbf{R}^n = \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$ $(n = n_1 + n_2)$, we write $x = (x^{(1)}, x^{(2)})$, $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, \dots, x_{n_i}^{(i)}) \in \mathbf{R}^{n_i}$ $(i = 1, 2)$. Let $\mathbf{D}_i = \mathbf{R}_+^{n_i+1} = \{(x^{(i)}, t_i) \in \mathbf{R}^{n_i+1}\colon t_i > 0\}$ and $\mathbf{D} = \mathbf{D}_1 \times \mathbf{D}_2$. If $X = (x^{(1)}, t_1; x^{(2)}, t_2) \in \mathbf{D}$, we also write $X = (x, t)$, where $x = (x^{(1)}, x^{(2)})$, $t = (t_1, t_2)$.

(2.2) We consider a linear transformation P_i of $\mathbf{R}^{n_i}$ such that $(P_i x^{(i)}, x^{(i)}) \geq (x^{(i)}, x^{(i)})$ for all $x^{(i)}$, where $(\cdot,\cdot)$ denotes the ordinary inner product. Let $A_{t_i}^{(i)} = \exp(P_i(\log t_i))$ $(t_i > 0)$. For $x^{(i)} \neq 0$, $\rho^{(i)}(x^{(i)})$ denotes the unique t_i such that $|A_{t_i^{-1}}^{(i)} x^{(i)}| = 1$, where $|\cdot| = (\cdot,\cdot)^{1/2}$. Let $\rho^{(i)}(0) = 0$. (See Calderón and Torchinsky [**1**].)

(2.3) Let $B^{(i)}$ be a subset of $\mathbf{R}^{n_i}$ such that $B^{(i)} = \{y^{(i)}\colon \rho^{(i)}(x^{(i)} - y^{(i)}) < t_i\}$ for some $x^{(i)} \in \mathbf{R}^{n_i}$ and $t_i > 0$. Then we say that $B^{(i)}$ is a ball centered at $x^{(i)}$ with radius t_i and write $B^{(i)} = B^{(i)}(x^{(i)}, t_i)$. We define $B_*^{(i)} = B_*^{(i)}(x^{(i)}, t_i) = B^{(i)}(x^{(i)}, 5t_i)$, $B_+^{(i)} = B_+^{(i)}(x^{(i)}, t_i) = B^{(i)}(x^{(i)}, 20t_i)$ and if $R = B^{(1)} \times B^{(2)}$, we set $R_* = B_*^{(1)} \times B_*^{(2)}$, $R_+ = B_+^{(1)} \times B_+^{(2)}$. We also write $R = R(x,t)$, $R_* = R_*(x,t)$, $R_+ = R_+(x,t)$.

3. Atomic decomposition. Let $\psi^{(i)} \in \mathscr{S}(\mathbf{R}^{n_i})$ (the Schwartz class) be such that

$$\operatorname{supp} \psi^{(i)} \subset B^{(i)}(0,1); \tag{3.1}$$

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$$\int_0^\infty \{\mathscr{F}\psi^{(i)}(A_{(t_i)}^{(i)^*}\xi^{(i)})\}^2 \frac{dt_i}{t_i} = 1 \quad \text{if } \xi^{(i)} \neq 0, \tag{3.2}$$

where $\mathscr{F}\psi^{(i)}$ is the Fourier transform of $\psi^{(i)}$ and $A_{t_i}^{(i)^*}$ is the transposed transformation of $A_{t_i}^{(i)}$;

$$\text{for every monomial } x^{(i)^{\alpha_i}} \text{ of degree } |\alpha_i| \leq N_i - 1, \quad \int \psi^{(i)}(x^{(i)}) x^{(i)^{\alpha_i}}\, dx^{(i)} = 0, \tag{3.3}$$

where N_i (≥ 1) will be determined later.

Let f be a tempered distribution on $\mathbf{R}^n$. Then the Lusin function for f is defined by

$$S(f) = \left(\int_{\Gamma(x)} |f * \psi_t(y)|^2 t_1^{-\gamma_1} t_2^{-\gamma_2}\, dy \frac{dt}{t_1 t_2} \right)^{1/2},$$

where

$$\Gamma(x) = \{(y,t) \in \mathbf{D}\colon\ \rho^{(i)}(x^{(i)} - y^{(i)}) < t_i \ (i = 1,2)\}$$

and

$$\psi_t(y) = \psi_{t_1}^{(1)}(y^{(1)}) \psi_{t_2}^{(2)}(y^{(2)})$$

with the usual notation: $\psi_{t_i}^{(i)}(y^{(i)}) = t_i^{-\gamma_i} \psi^{(i)}(A_{t_i}^{(i)^{-1}} y^{(i)})$, $\gamma_i = \operatorname{trace} P_i$. Let $\varphi^{(i)} \in \mathscr{S}(\mathbf{R}^{n_i})$ be such that $\operatorname{supp} \varphi^{(i)} \subset B^{(i)}(0,1)$, $\int \varphi^{(i)}\, dx^{(i)} = 1$. Set $\varphi_t(x) = \varphi_{t_1}^{(1)}(x^{(1)}) \varphi_{t_2}^{(2)}(x^{(2)})$ and define the radial maximal function f^+ by $f^+(x) = \sup_t |f * \varphi_t(x)|$. Then it is known that if $f^+ \in L^p$ $(0 < p < \infty)$, then $S(f) \in L^p$ and $||S(f)||_p \leq c||f^+||_p$ (see [**7**]).

In this section we prove that if $f \in L^2$ (this is assumed for simplicity) and $S(f) \in L^p$ $(0 < p \leq 1)$, then f can be decomposed into atoms, showing at the same time the inequality: $||f^+||_p \leq c||S(f)||_p$. Here an atom is a function a vanishing outside an open set U of finite measure and satisfying

$$|a * \varphi_t(x)| \leq \left(\frac{|R_+(x,t) \cap U|}{|R(x,t)|} \right)^\alpha M(x) \quad \text{for all } x \text{ and } t,$$

where $\alpha > (2-p)/2p$ and M is a function such that

$$||M||_2 \leq |U|^{1/2 - 1/p}$$

(for $R(x,t)$, $R_+(x,t)$ see (2.3)).

THEOREM 1. *Suppose that $f \in L^2$ and $S(f) \in L^p$ $(0 < p \leq 1)$. Then there are a sequence of numbers $\{\lambda_j\}_{j=1}^\infty$ and a sequence of atoms $\{a_j\}_{j=1}^\infty$ such that*

(a) $f = \sum_{j=1}^\infty \lambda_j a_j$ *(the series converges in $\mathscr{S}'$)*;

(b) $\sum_{j=1}^\infty |\lambda_j|^p \leq c||S(f)||_p^p$, *where c is independent of f.*

As a consequence we have

THEOREM 2. *Let f be a tempered distribution on $\mathbf{R}^n$ such that*

$$\mathscr{F}f(\xi)(1 + |\xi|^2)^{-l} \in L^2(\mathbf{R}^n)$$

for some $l \geq 0$. *Then*

$$||f^+||_p \leq c||S(f)||_p \qquad (0 < p \leq 1).$$

For the case when P_i is diagonal, see [**7**].

Now we prove Theorem 1. We first require some preliminaries. Following [**5**], let $\{B^{(i)}(x_j^{(i)}, 2^{-1})\colon j = 1, 2, \dots\}$ be a maximal family of mutually disjoint balls. Let $B_{k,j}^{(i)} = B^{(i)}(A_{2^k}^{(i)} x_j^{(i)}, 2^k)$ for an integer k and set $\mathscr{B}(i,k) = \{B_{k,j}^{(i)}\colon j = 1, 2, \dots\}$, $\mathscr{B}(i) = \bigcup_{k=-\infty}^{\infty} \mathscr{B}(i,k)$. Then we can easily see

(3.4) REMARK. Similar results to those of [**5**, Lemma 7.14] hold for $\mathscr{B}(i)$.

This remark enables us to proceed as in [**5**, Chapter 7]. For $B^{(i)} \in \mathscr{B}(i,k)$, let $\varsigma_{B^{(i)}}^{(i)} = \chi_{B^{(i)}} / \sum_{C^{(i)} \in \mathscr{B}(i,k)} \chi_{C^{(i)}}$ and $I_{B^{(i)}}^{(i)} = \{t_i \in \mathbf{R}\colon 2^{k+1} < t_i \leq 2^{k+2}\}$. Set $\mathscr{R} = \{B^{(1)} \times B^{(2)}\colon B^{(i)} \in \mathscr{B}(i)\ (i = 1, 2)\}$. For $R = B^{(1)} \times B^{(2)} \in \mathscr{R}$, let $\varsigma_R(x) = \varsigma_{B^{(1)}}^{(1)}(x^{(1)}) \varsigma_{B^{(2)}}^{(2)}(x^{(2)})$, $I_R = I_{B^{(1)}}^{(1)} \times I_{B^{(2)}}^{(2)}$, $D(R) = R \times I_R$ and set

$$b_R(x) = \int_{D(R)} \varsigma_R(y) f * \psi_t(y) \psi_t(x-y)\, dy \, \frac{dt}{t_1 t_2},$$

$$S_R = \left(\int_{D(R)} |f * \psi_t(y)|^2 \, dy \, \frac{dt}{t_1 t_2} \right)^{1/2}.$$

Next for an integer k, let $O_k = \{x \in \mathbf{R}^n\colon S(f)(x) > 2^k\}$, $\mathscr{R}(k) = \{R \in \mathscr{R}\colon |R \cap O_k| \geq |R|/2,\ |R \cap O_{k+1}| < |R|/2\}$. Then as in [**2** and **5**], we have

$$\sum_{R \in \mathscr{R}} b_R = f \quad \text{in } \mathscr{S}'; \tag{3.5}$$

$$\sum_{R \in \mathscr{R}(k)} S_R^2 \leq c 2^{2k} |O_k| \tag{3.6}$$

(3.7) if $\mathscr{R}'$ is a subset of $\mathscr{R}$, then

$$\left\| \sum_{R \in \mathscr{R}'} b_R \right\|_2^2 \leq c \sum_{R \in \mathscr{R}'} S_R^2.$$

Let $a_k = \sum_{R \in \mathscr{R}(k)} b_R$, $U_k = \{x \in \mathbf{R}^n\colon M_S(\chi_{O_k})(x) \geq 100^{-\gamma_1 - \gamma_2}\}$. Here M_S is the strong maximal operator defined by $M_S(f)(x) = \sup_{x \in R} |R|^{-1} \int_R |f(y)|\, dy$, where $R = B^{(1)} \times B^{(2)}$ and $B^{(i)}$ is a ball in $\mathbf{R}^{n_i}$. Suppose that N_1 and N_2 in (3.3) are large enough to satisfy $\delta = \delta(N_1, N_2) = \min\{N_1/\gamma_1, N_2/\gamma_2\} - 1 > (2-p)/2p$. Then, as in [**4**], to prove Theorem 1 it is sufficient to show

$$|a_k * \varphi_t(x)| \leq \left(\frac{|R_+(x,t) \cap U_k|}{|R(x,t)|} \right)^{\delta} L(x), \tag{3.8}$$

where L is a function such that $||L||_2^2 \leq c 2^{2k} |O_k|$.

To prove (3.8), fix k, x, t and let $\mathscr{R}^* = \{R \in \mathscr{R}(k)\colon R_* \cap S \text{ is not empty}\}$, where $S = R(x,t)$. Then since $\operatorname{supp} b_R \subset R_*$, we have $a_k * \varphi_t(x) = \sum_{R \in \mathscr{R}^*} b_R * \varphi_t(x)$.

If $R = B^{(1)}(y^{(1)}, s_1) \times B^{(2)}(y^{(2)}, s_2)$, then we define $l_1(R) = s_1$, $l_2(R) = s_2$. With this notation we classify $\mathscr{R}^*$ as follows:

$$\begin{aligned}
\mathscr{L}(1) &= \{R \in \mathscr{R}^*\colon\ l_1(R) < l_1(S),\ l_2(R) < l_2(S)\},\\
\mathscr{L}(2) &= \{R \in \mathscr{R}^*\colon\ l_1(R) < l_1(S),\ l_2(R) \geq l_2(S)\},\\
\mathscr{L}(3) &= \{R \in \mathscr{R}^*\colon\ l_1(R) \geq l_1(S),\ l_2(R) < l_2(S)\},\\
\mathscr{L}(4) &= \{R \in \mathscr{R}^*\colon\ l_1(R) \geq l_1(S),\ l_2(R) \geq l_2(S)\}.
\end{aligned}$$

For $j = 1, 2, 3, 4$, let $A_j = \sum_{R \in \mathscr{L}(j)} b_R$ (if $\mathscr{L}(j)$ is empty, let $A_j = 0$. This rule is in effect throughout this note). We estimate $A_j * \varphi_t(x)$ separately and prove a similar estimate to (3.8) for each A_j.

First we estimate $A_1 * \varphi_t(x)$. It is sufficient to prove

$$|A_1 * \varphi_t(x)| \leq c \left(\frac{|S_+ \cap U_k|}{|S|}\right)^{\delta} \left\{ \sum_{R \in \mathscr{R}(k)} M_S(b_R)(x)^2 \right\}^{1/2}, \tag{3.9}$$

since by (3.6) and (3.7) we have

$$\int \sum_{R \in \mathscr{R}(k)} M_S(b_R)^2\, dx \leq c \sum_{R \in \mathscr{R}(k)} \int |b_R|^2\, dx \leq c \sum_{R \in \mathscr{R}(k)} S_R^2 \leq c 2^{2k} |O_k|.$$

Let $R \in \mathscr{L}(1)$. We estimate $b_R * \varphi_t(x)$. Recall that $\operatorname{supp} b_R \subset R_*$. Therefore, since $\psi^{(i)}$ has vanishing moments up to the order $N_i - 1$, using Taylor's formula and noting $R_* \subset S_+$, we easily find

$$|b_R * \varphi_t(x)| \leq c \left(\frac{l_1(R)}{t_1}\right)^{N_1} \left(\frac{l_2(R)}{t_2}\right)^{N_2} |S|^{-1} \int_{S_+} |b_R|\, dy.$$

Next, since $R \subset S_+$, it follows that $|S_+ \cap U_k| \geq |R|/2$. Combining these inequalities, we have

$$|b_R * \varphi_t(x)| \leq c(|S_+ \cap U_k|/|S|)^{\delta} |R|\, |S|^{-1} M_S(b_R)(x).$$

Thus

$$|A_1 * \varphi_t(x)| \leq c \sum_{R \in \mathscr{L}(1)} \left(\frac{|S_+ \cap U_k|}{|S|}\right)^{\delta} |R|\, |S|^{-1} M_S(b_R)(x). \tag{3.10}$$

Let $\mathscr{K}(S) = \{R \in \mathscr{R}\colon\ R \subset S_+\}$. Then, note that (3.4) implies

$$\sum_{R \in \mathscr{L}(1)} \left(\frac{|R|}{|S|}\right)^2 \leq \sum_{R \in \mathscr{K}(S)} \left(\frac{|R|}{|S|}\right)^2 \leq c,$$

where c is independent of x and t. Thus, applying the Schwarz inequality to the right hand side of (3.10), we obtain (3.9).

Next we estimate $A_2 * \varphi_t(x)$. For an integer m let $\mathscr{M}(m) = \{R \in \mathscr{L}(2)\colon\ l_1(R) = 2^m\}$ and $A_{2,m} = \sum_{R \in \mathscr{M}(m)} b_R$. Then $A_2 = \sum_{m=-\infty}^{\infty} A_{2,m}$ (recall that $A_{2,m} = 0$ if

$\mathscr{M}(m)$ is empty). Let

$$\mathscr{C}(m) = \{B^{(1)} \in \mathscr{B}(1)\colon\ B^{(1)} \times B^{(2)} \in \mathscr{M}(m) \text{ for some } B^{(2)} \in \mathscr{B}(2)\},$$
$$\mathscr{D}(m, B^{(1)}) = \{R \in \mathscr{M}(m)\colon\ R = B^{(1)} \times B^{(2)} \text{ for some } B^{(2)} \in \mathscr{B}(2)\},$$
$$\mathscr{D}(B^{(1)}) = \{R \in \mathscr{R}(k)\colon\ R = B^{(1)} \times B^{(2)} \text{ for some } B^{(2)} \in \mathscr{B}(2)\} \qquad (B^{(1)} \in \mathscr{B}(1)),$$
$$\mathscr{E}(B^{(1)}) = \mathscr{D}(B^{(1)}) \cap \mathscr{L}(1).$$

Then

$$\begin{aligned} A_{2,m} * \varphi_t(x) &= \sum_{B^{(1)} \in \mathscr{C}(m)} \int \varphi_t(x-y) \sum_{R \in \mathscr{D}(m, B^{(1)})} b_R(y)\, dy \\ &= \sum_{B^{(1)} \in \mathscr{C}(m)} \int \varphi_t(x-y) \sum_{R \in \mathscr{D}(B^{(1)})} b_R(y)\, dy \\ &\quad - \sum_{B^{(1)} \in \mathscr{C}(m)} \int \varphi_t(x-y) \sum_{R \in \mathscr{E}(B^{(1)})} b_R(y)\, dy \\ &= H_m - I_m, \quad \text{say.} \end{aligned}$$

As in the estimate for $A_1 * \varphi_t(x)$, we have

$$\left| \sum_{m=-\infty}^{\infty} I_m \right| \le c \left(\frac{|S_+ \cap U_k|}{|S|} \right)^{\delta} \left\{ \sum_{R \in \mathscr{R}(k)} M_S(b_R)(x)^2 \right\}^{1/2}. \tag{3.11}$$

It remains to estimate $\sum_m H_m$. Let $G_{B^{(1)}}(y) = \sum_{R \in \mathscr{D}(B^{(1)})} b_R(y)$ $(B^{(1)} \in \mathscr{B}(1))$. Then, for $B^{(1)} \in \mathscr{C}(m)$, consider the integral:

$$J = \int \varphi_t(x^{(1)} - y^{(1)}, x^{(2)} - y^{(2)}) G_{B^{(1)}}(y^{(1)}, y^{(2)})\, dy^{(1)}.$$

If we fix $y^{(2)}$ and regard $G_{B^{(1)}}$ as a function of $y^{(1)}$, then $G_{B^{(1)}}$ has vanishing moments up to the order $N_1 - 1$ and is supported in $B_*^{(1)}$. By Taylor's formula, this implies that

$$|J| \le c t_1^{-\gamma_1} t_2^{-\gamma_2} \left(\frac{2^m}{t_1} \right)^{N_1} \int_{B_*^{(1)}} |G_{B^{(1)}}(y^{(1)}, y^{(2)})|\, dy^{(1)}. \tag{3.12}$$

Note that if $B^{(1)} \in \mathscr{C}(m)$, then $B_*^{(1)} \subset B^{(1)}(x^{(1)}, 20t_1)$. Thus, integrating J with respect to $y^{(2)}$ and using (3.12), we find

$$\left| \int \varphi_t(x-y) G_{B^{(1)}}(y)\, dy \right| \le c \left(\frac{2^m}{t_1} \right)^{N_1} |S|^{-1} \int_{S_+} |G_{B^{(1)}}(y)|\, dy.$$

Consequently

$$|H_m| \le c \left(\frac{2^m}{t_1} \right)^{N_1} |S|^{-1} \int_{S_+} \sum_{B^{(1)} \in \mathscr{C}(m)} |G_{B^{(1)}}(y)|\, dy. \tag{3.13}$$

Let $F_m(y) = \sum_{B^{(1)}\in\mathscr{B}(1,m)} |G_{B^{(1)}}(y)|^2$. Since by Remark (3.4) we have

$$\sum_{B^{(1)}\in\mathscr{B}(1,m)} \chi_{B_*^{(1)}} \le c$$

(where c is independent of m), from the Schwarz inequality it follows that $\sum_{B^{(1)}\in\mathscr{C}(m)} |G_{B^{(1)}}| \le cF_m^{1/2}$. Therefore, by (3.13) we have

$$|H_m| \le c(2^m/t_1)^{N_1} M_S(F_m^{1/2}),$$

so that

$$\left|\sum_m H_m\right| \le c\sum_m \left(\frac{2^m}{t_1}\right)^{N_1} M_S(F_m^{1/2}), \tag{3.14}$$

where the summation with respect to m is taken over all integers m such that $\mathscr{M}(m)$ is not empty. Note that if $\mathscr{M}(m)$ is not empty,

$$(2^m/t_1)^{\gamma_1} \le c|S_+ \cap U_k|/|S|.$$

Thus, applying the Schwarz inequality to the right-hand side of (3.14), we have

$$\left|\sum_m H_m\right| \le c\left(\frac{|S_+\cap U_k|}{|S|}\right)^{N_1/\gamma_1} \left\{\sum_{m=-\infty}^{\infty} M_S(F_m^{1/2})^2\right\}^{1/2}. \tag{3.15}$$

On the other hand, by (3.6) and (3.7)

$$\begin{aligned}\int \sum_{m=-\infty}^{\infty} M_S(F_m^{1/2})^2\,dx &\le c\sum_{m=-\infty}^{\infty}\int F_m\,dx \le c\sum_{m=-\infty}^{\infty}\sum_{B^{(1)}\in\mathscr{B}(1,m)}\sum_{R\in\mathscr{D}(B^{(1)})} S_R^2\\ &\le c2^{2k}|O_k|.\end{aligned}$$

Therefore, combining (3.15) with (3.11), we obtain a desired estimate. $A_3 * \varphi_t(x)$ can be treated similarly.

Finally we estimate $A_4 * \varphi_t(x)$. Note that if $\mathscr{L}(4)$ is not empty, then $U_k \supset S_+$. Thus

$$\begin{aligned}|A_4 * \varphi_t(x)| &\le \sum_{j=1}^{3} |A_j * \varphi_t(x)| + cM_S(a_k)(x)\\ &\le \sum_{j=1}^{3} |A_j * \varphi_t(x)| + c\left(\frac{|S_+\cap U_k}{|S|}\right)^{\delta} M_S(a_k).\end{aligned}$$

This is what we need since we have already obtained desired estimates for $A_j * \varphi_t(x)$ $(j = 1, 2, 3)$ and by (3.6) and (3.7) we have $||a_k||_2^2 \le c2^{2k}|O_k|$. This completes a proof of (3.8).

4. Double singular integrals. Let $K^{(i)} \in C^\infty(\mathbf{R}^{n_i} - \{0\})$ be such that

$$\int_{|x^{(i)}|=1} K^{(i)}(x^{(i)})(P_i x^{(i)}, x^{(i)})\,d\sigma(x^{(i)}) = 0,$$

$$K^{(i)}(A_{t_i}^{(i)} x^{(i)}) = t_i^{-\gamma_i} K^{(i)}(x^{(i)}) \quad \text{for all } t_i > 0,$$

where $d\sigma(x^{(i)})$ is the area element of $S^{n_i-1} = \{x^{(i)}: |x^{(i)}| = 1\}$. For $\varepsilon_1, \varepsilon_2 > 0$, set

$$K_{\varepsilon_1,\varepsilon_2}(x) = \prod_{i=1,2} K^{(i)}(x^{(i)})\{1 - \chi_{[0,1]}(\varepsilon_i^{-1}\rho^{(i)}(x^{(i)}))\}.$$

We can generalize the weak type estimates of [**3** and **7**].

THEOREM 3. *Let A and B be compact sets of $\mathbf{R}^n$. If f is a function on $\mathbf{R}^n$ such that $\int_B |f|\log(2+|f|)\,dx < \infty$ and $\operatorname{supp} f \subset B$, then*

$$\left|\left\{x \in A: \sup_{\varepsilon_1,\varepsilon_2>0} |f * K_{\varepsilon_1,\varepsilon_2}(x)| > 1\right\}\right| \le c\int_B |f|\log(2+|f|)\,dx,$$

where c is a constant independent of f.

Combining results of [**7**] with Theorem 2, we obtain the equivalence with respect to the L^p-norms of Lusin functions and radial maximal functions (radial maximal functions can be replaced by nontangential maximal functions). Now we briefly see how this equivalence implies Theorem 3. Let $h \in C^\infty(\mathbf{R}^1)$ be a nonnegative function such that $h(u) = 1$ if $u \le 1$ and $h(u) = 0$ if $u \ge 2$. Set

$$K'_{\varepsilon_1,\varepsilon_2}(x) = K_{\varepsilon_1,\varepsilon_2}(x) \prod_{i=1,2} h(M\rho^{(i)}(x^{(i)})),$$

where $M > 0$. Then, by Stein's theorem on limits of sequences of operators, to prove Theorem 3 it is sufficient to show that

$$\sup_{\varepsilon_1,\varepsilon_2>0} |f * K'_{\varepsilon_1,\varepsilon_2}(x)| < \infty \quad \text{for almost every } x, \tag{4.1}$$

where f is a function with compact support such that $\int |f|\log(2+|f|)\,dx < \infty$, $\int f(x^{(1)}, x^{(2)})\,dx^{(2)} = 0$ for all $x^{(1)} \in \mathbf{R}^{n_1}$, $\int f(x^{(1)}, x^{(2)})\,dx^{(1)} = 0$ for all $x^{(2)} \in \mathbf{R}^{n_2}$. We can show by a direct estimate that $f^+ \in L^p$ for some $p < 1$. Thus, as in [**7**], (4.1) follows from the inequality

$$\sup_{\delta>0} ||(f * K^{(\delta)})^+||_p \le c||f^+||_p, \tag{4.2}$$

where

$$K^{(\delta)}(x) = \prod_{i=1,2} K^{(i)}(x^{(i)})h(M\rho^{(i)}(x^{(i)}))\{1 - h(\delta^{-1}\rho^{(i)}(x^{(i)}))\}.$$

(4.2) is proved as follows. First, by Theorem 2 we have that $||(f * K^{(\delta)})^+||_p \le c||S(f * K^{(\delta)})||_p$. Next, it is not difficult to see that $\sup_{\delta>0} ||S(f * K^{(\delta)})||_p \le c||S(f)||_p$. Finally, from results of [**7**] it follows that $||S(f)||_p \le c||f^+||_p$. Combining these results, we obtain (4.2). See [**7**] for more details.

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MATHEMATICAL INSTITUTE, TÔHOKU UNIVERSITY, SENDAI, 980, JAPAN