AN ATOMIC DECOMPOSITION FOR PARABOLIC $H^p$ SPACES ON PRODUCT DOMAINS

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ABSTRACT. We obtain an atomic decomposition for two-parameter parabolic $H^p$ spaces, showing simultaneously an integral inequality between Lusin functions and nontangential maximal functions. As its consequence, we generalize Fefferman's weak type estimates for double singular integrals.

1. Introduction. Gundy and Stein [6] proved that $H^p$ spaces on the bidisc are characterized in terms of Lusin functions. Under a certain restriction, this was extended to parabolic $H^p$ spaces by [7]. In the present note we consider parabolic $H^p$ spaces on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ again and extend R. Fefferman's atomic decomposition (see [4]) to this setting, which enables us to remove the restriction mentioned above and brings the Lusin function characterization of two-parameter parabolic $H^p$ spaces. (See Theorems 1 and 2.) As a consequence, C. Fefferman's weak type estimates for double singular integrals are further generalized (see [3, 7] and Theorem 3).

2. Preliminaries. Here are some of our notations and background materials.

(2.1) If $x \in \mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ ($n = n_1 + n_2$), we write $x = (x^{(1)}, x^{(2)})$, $x^{(i)} = (x^{(i)}_1, x^{(i)}_2, \ldots, x^{(i)}_{n_i}) \in \mathbb{R}^{n_i}$ ($i = 1, 2$). Let $D_i = \mathbb{R}^{n_i+1} = \{(x^{(i)}_t) \in \mathbb{R}^{n_i+1} : t > 0\}$ and $D = D_1 \times D_2$. If $X = (x^{(1)}, t_1; x^{(2)}, t_2) \in D$, we also write $X = (x, t)$, where $x = (x^{(1)}, x^{(2)})$, $t = (t_1, t_2)$.

(2.2) We consider a linear transformation $P_i$ of $\mathbb{R}^{n_i}$ such that $(P_i x^{(i)}, x^{(i)}) > (x^{(i)}, x^{(i)})$ for all $x^{(i)}$, where $(\cdot, \cdot)$ denotes the ordinary inner product. Let $A_i(t) = \exp(P_i(\log t_i))$ ($t_i > 0$). For $x^{(i)} \neq 0$, $\rho^{(i)}(x^{(i)})$ denotes the unique $t_i$ such that $|A_i^{-1} x^{(i)}| = 1$, where $|\cdot| = (\cdot, \cdot)^{1/2}$. Let $\rho^{(i)}(0) = 0$. (See Calderón and Torchinsky [1].)

(2.3) Let $B^{(i)}$ be a subset of $\mathbb{R}^{n_i}$ such that $B^{(i)} = \{y^{(i)} : \rho^{(i)}(x^{(i)} - y^{(i)}) < t_i\}$ for some $x^{(i)} \in \mathbb{R}^{n_i}$ and $t_i > 0$. Then we say that $B^{(i)}$ is a ball centered at $x^{(i)}$ with radius $t_i$ and write $B^{(i)} = B^{(i)}(x^{(i)}, t_i)$. We define $B^{(i)}_* = B^{(i)}(x^{(i)}, t_i) = B^{(i)}(x^{(i)}_1, 5t_i)$, $B^{(i)}_+ = B^{(i)}(x^{(i)}, t_i) = B^{(i)}(x^{(i)}_1, 20t_i)$ and if $R = B^{(1)} \times B^{(2)}$, we set $R_* = B^{(1)}_* \times B^{(2)}_*$, $R_+ = B^{(1)}_+ \times B^{(2)}_+$. We also write $R = R(x, t)$, $R_* = R_*(x, t)$, $R_+ = R_+(x, t)$.

3. Atomic decomposition. Let $\psi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ (the Schwartz class) be such that

\begin{equation}
\text{supp} \psi^{(i)} \subset B^{(i)}(0, 1);
\end{equation}

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(3.2) \[ \int_0^\infty \{ \mathcal{F} \psi^{(i)}(A_{(i)}^{(i)} \xi^{(i)}) \}^2 \frac{dt_i}{t_i} = 1 \] if \( \xi^{(i)} \neq 0 \),

where \( \mathcal{F} \psi^{(i)} \) is the Fourier transform of \( \psi^{(i)} \) and \( A_{(i)}^{(i)} \) is the transposed transformation of \( A_{(i)}^{(i)} \);

(3.3) for every monomial \( x^{(i)\alpha_i} \) of degree \( |\alpha_i| \leq N_i - 1 \),

\[ \int \psi^{(i)}(x^{(i)})x^{(i)\alpha_i} dx^{(i)} = 0, \]

where \( N_i (\geq 1) \) will be determined later.

Let \( f \) be a tempered distribution on \( \mathbb{R}^n \). Then the Lusin function for \( f \) is defined by

\[ S(f) = \left( \int_{\Gamma(x)} |f * \varphi_t(y)|^2 t_1^{-\gamma_1} t_2^{-\gamma_2} dy \frac{dt}{t_1 t_2} \right)^{1/2}, \]

where

\[ \Gamma(x) = \{(y, t) \in D: \rho^{(i)}(x^{(i)} - y^{(i)}) < t_i \ (i = 1, 2)\} \]

and

\[ \psi_t(y) = \psi_{t_1}^{(1)}(y^{(1)}) \psi_{t_2}^{(2)}(y^{(2)}) \]

with the usual notation: \( \psi_{t_i}^{(i)}(y^{(i)}) = t_i^{-\gamma_i} \psi^{(i)}(A_{(i)}^{(i)-1} y^{(i)}) \), \( \gamma_i = \text{trace } P_i \). Let \( \varphi^{(i)} \in \mathcal{P} (\mathbb{R}^{n_i}) \) be such that \( \text{supp } \varphi^{(i)} \subset B^{(i)}(0,1) \), \( \int \varphi^{(i)} dx^{(i)} = 1 \). Set \( \varphi_t(x) = \varphi_{t_1}^{(1)}(x^{(1)}) \varphi_{t_2}^{(2)}(x^{(2)}) \) and define the radial maximal function \( f^+ \) by \( f^+(x) = \sup_t |f * \varphi_t(x)| \). Then it is known that if \( f^+ \in L^p \ (0 < p < \infty) \), then \( S(f) \in L^p \) and \( \|S(f)\|_p \leq c\|f^+\|_p \) (see [7]).

In this section we prove that if \( f \in L^2 \) (this is assumed for simplicity) and \( S(f) \in L^p \ (0 < p \leq 1) \), then \( f \) can be decomposed into atoms, showing at the same time the inequality: \( \|f^+\|_p \leq c\|S(f)\|_p \). Here an atom is a function \( a \) vanishing outside an open set \( U \) of finite measure and satisfying

\[ |a * \varphi_t(x)| \leq \left( \frac{|R_+(x,t) \cap U|}{|R(x,t)|} \right)^\alpha M(x) \] for all \( x \) and \( t \),

where \( \alpha > (2-p)/2p \) and \( M \) is a function such that

\[ \|M\|_2 \leq |U|^{1/2-1/p} \]

(for \( R(x,t), R_+(x,t) \) see (2.3)).

**Theorem 1.** Suppose that \( f \in L^2 \) and \( S(f) \in L^p \ (0 < p \leq 1) \). Then there are a sequence of numbers \( \{\lambda_j\}_{j=1}^\infty \) and a sequence of atoms \( \{a_j\}_{j=1}^\infty \) such that

(a) \( f = \sum_{j=1}^\infty \lambda_j a_j \) (the series converges in \( \mathcal{S}' \));

(b) \( \sum_{j=1}^\infty |\lambda_j|^p \leq c\|S(f)\|_p^p \), where \( c \) is independent of \( f \).

As a consequence we have

**Theorem 2.** Let \( f \) be a tempered distribution on \( \mathbb{R}^n \) such that

\[ \mathcal{F} f(\xi)(1 + |\xi|^2)^{-t} \in L^2(\mathbb{R}^n) \]
for some \( l \geq 0 \). Then
\[
\| f^+ \|_p \leq c \| S(f) \|_p \quad (0 < p \leq 1).
\]

For the case when \( P_t \) is diagonal, see [7].

Now we prove Theorem 1. We first require some preliminaries. Following [5], let \( \{ B^{(i)}(x_j^{(i)}, 2^{-1}): j = 1, 2, \ldots \} \) be a maximal family of mutually disjoint balls. Let \( B_{k,j}^{(i)} = B^{(i)}(A_{2k} x_j^{(i)}, 2^k) \) for an integer \( k \) and set \( \mathcal{B}(i, k) = \{ B_{k,j}^{(i)}: j = 1, 2, \ldots \} \), \( \mathcal{B}(i) = \bigcup_{k=-\infty}^{\infty} \mathcal{B}(i, k) \). Then we can easily see (3.4) REMARK. Similar results to those of [5, Lemma 7.14] hold for \( \mathcal{B}(i) \).

This remark enables us to proceed as in [5, Chapter 7]. For \( B^{(i)} \in \mathcal{B}(i, k) \), let \( \xi^{(i)}_{B^{(i)}} = \chi_{B^{(i)}} / \sum_{C^{(i)} \in \mathcal{B}(i, k)} \chi_{C^{(i)}} \) and \( I_{B^{(i)}}^{(i)} = \{ t_i \in \mathbb{R}: 2^{k+1} < t_i \leq 2^{k+2} \} \). Set \( \mathcal{F} = \{ B^{(1)} \times B^{(2)}: B^{(i)} \in \mathcal{B}(i) \ (i = 1, 2) \} \). For \( R = B^{(1)} \times B^{(2)} \in \mathcal{F} \), let \( \zeta_R(x) = \xi_{B^{(1)}}(x_1) \xi_{B^{(2)}}(x_2) \), \( I_R = I_{B^{(1)}}^{(1)} \times I_{B^{(2)}}^{(2)} \), \( D(R) = R \times I_R \) and set
\[
b_R(x) = \int_{D(R)} \zeta_R(y) f^*(y) \psi_t(x-y) \, dy \, \frac{dt}{t_1 t_2},
\]
\[
S_R = \left( \int_{D(R)} |f^* \psi_t(y)|^2 \, dy \, \frac{dt}{t_1 t_2} \right)^{1/2}.
\]

Next for an integer \( k \), let \( O_k = \{ x \in \mathbb{R}^n: S(f)(x) > 1 \} \), \( \mathcal{R}(k) = \{ R \in \mathcal{F}: |R \cap O_k| \geq |R|/2, |R \cap O_{k+1}| < |R|/2 \} \). Then as in [2 and 5], we have
\[
(3.5) \quad \sum_{R \in \mathcal{R}} b_R = f \quad \text{in \( \mathcal{F} \)},
\]
\[
(3.6) \quad \sum_{R \in \mathcal{R}(k)} S_R^2 \leq c 2^{2k} |O_k|
\]
\[
(3.7) \quad \text{if \( \mathcal{R}' \) is a subset of \( \mathcal{R} \), then}
\]
\[
\left\| \sum_{R \in \mathcal{R}'} b_R \right\|_2^2 \leq c \sum_{R \in \mathcal{R}'} S_R^2.
\]

Let \( a_k = \sum_{R \in \mathcal{R}(k)} b_R \), \( U_k = \{ x \in \mathbb{R}^n: M_S(\chi_{O_k})(x) \geq 100^{-N_1-N_2} \} \}. \) Here \( M_S \) is the strong maximal operator defined by \( M_S(f)(x) = \sup_{x \in R} |R|^{-1} \int_R |f(y)| \, dy \), where \( R = B^{(1)} \times B^{(2)} \) and \( B^{(i)} \) is a ball in \( \mathbb{R}^n \). Suppose that \( N_1 \) and \( N_2 \) in (3.3) are large enough to satisfy \( \delta = \delta(N_1, N_2) = \min\{N_1/\gamma_1, N_2/\gamma_2\} - 1 > (2-p)/2p \). Then, as in [4], to prove Theorem 1 it is sufficient to show
\[
(3.8) \quad |a_k \ast \varphi_t(x)| \leq \left( \frac{|R_+(x, t) \cap U_k|}{|R(x, t)|} \right)^{\delta} L(x),
\]
where \( L \) is a function such that \( \|L\|_2^2 \leq c 2^{2k} |O_k| \).

To prove (3.8), fix \( k, x, t \) and let \( \mathcal{R}' = \{ R \in \mathcal{R}(k): R \cap S \) is not empty \} \), where \( S = R(x, t) \). Then since \( \sup b_R \subset R_+ \), we have \( a_k \ast \varphi_t(x) = \sum_{R \in \mathcal{R}'} b_R \ast \varphi_t(x) \).
If $R = B^{(1)}(y^{(1)}, s_1) \times B^{(2)}(y^{(2)}, s_2)$, then we define $l_1(R) = s_1$, $l_2(R) = s_2$. With this notation we classify $\mathcal{R}^*$ as follows:

- $\mathcal{L}(1) = \{R \in \mathcal{R}^*: l_1(R) < l_1(S), l_2(R) < l_2(S)\}$,
- $\mathcal{L}(2) = \{R \in \mathcal{R}^*: l_1(R) < l_1(S), l_2(R) \geq l_2(S)\}$,
- $\mathcal{L}(3) = \{R \in \mathcal{R}^*: l_1(R) \geq l_1(S), l_2(R) < l_2(S)\}$,
- $\mathcal{L}(4) = \{R \in \mathcal{R}^*: l_1(R) \geq l_1(S), l_2(R) \geq l_2(S)\}$.

For $j = 1, 2, 3, 4$, let $A_j = \sum_{R \in \mathcal{L}(j)} b_R$ (if $\mathcal{L}(j)$ is empty, let $A_j = 0$. This rule is in effect throughout this note). We estimate $A_j * \varphi_t(x)$ separately and prove a similar estimate to (3.8) for each $A_j$.

First we estimate $A_1 * \varphi_t(x)$. It is sufficient to prove

\[
|A_1 * \varphi_t(x)| \leq c \left( \frac{|S_+ \cap U_k|}{|S|} \right)^{\frac{1}{2}} \sum_{R \in \mathcal{R}(k)} M_S(b_R)(x)^{\frac{1}{2}}.
\]

since by (3.6) and (3.7) we have

\[
\int \sum_{R \in \mathcal{R}(k)} M_S(b_R)^2 \, dx \leq c \sum_{R \in \mathcal{R}(k)} |b_R|^2 \, dx \leq c \sum_{R \in \mathcal{R}(k)} S_R^2 \leq c 2^{2k}|O_k|.
\]

Let $R \in \mathcal{L}(1)$. We estimate $b_R * \varphi_t(x)$. Recall that supp $b_R \subset R_*$. Therefore, since $\psi^{(t)}$ has vanishing moments up to the order $N_t - 1$, using Taylor's formula and noting $R_* \subset S_+$, we easily find

\[
|b_R * \varphi_t(x)| \leq c \left( \frac{l_1(R)}{t_1} \right)^{N_t} \left( \frac{l_2(R)}{t_2} \right)^{N_t} |S|^{-1} \int_{S_+} |b_R| \, dy.
\]

Next, since $R \subset S_+$, it follows that $|S_+ \cap U_k| \geq |R|/2$. Combining these inequalities, we have

\[
|b_R * \varphi_t(x)| \leq c (|S_+ \cap U_k|/|S|)^{\delta} |R| |S|^{-1} M_S(b_R)(x).
\]

Thus

\[
|A_1 * \varphi_t(x)| \leq c \sum_{R \in \mathcal{L}(1)} \left( \frac{|S_+ \cap U_k|}{|S|} \right)^{\delta} |R| |S|^{-1} M_S(b_R)(x).
\]

Let $\mathcal{R}(S) = \{R \in \mathcal{R}: R \subset S_+\}$. Then, note that (3.4) implies

\[
\sum_{R \in \mathcal{L}(1)} \left( \frac{|R|}{|S|} \right)^2 \leq \sum_{R \in \mathcal{R}(S)} \left( \frac{|R|}{|S|} \right)^2 \leq c,
\]

where $c$ is independent of $x$ and $t$. Thus, applying the Schwarz inequality to the right hand side of (3.10), we obtain (3.9).

Next we estimate $A_2 * \varphi_t(x)$. For an integer $m$ let $\mathcal{M}(m) = \{R \in \mathcal{L}(2): l_1(R) = 2^m\}$ and $A_{2,m} = \sum_{R \in \mathcal{M}(m)} b_R$. Then $A_2 = \sum_{m=-\infty}^{\infty} A_{2,m}$ (recall that $A_{2,m} = 0$ if
\( \mathcal{M}(m) \) is empty. Let
\[ \mathcal{E}(m) = \{B^{(1)} \in \mathcal{B}(1): B^{(1)} \times B^{(2)} \in \mathcal{M}(m) \text{ for some } B^{(2)} \in \mathcal{B}(2)\}, \]
\[ \mathcal{D}(m, B^{(1)}) = \{R \in \mathcal{M}(m): R = B^{(1)} \times B^{(2)} \text{ for some } B^{(2)} \in \mathcal{B}(2)\}, \]
\[ \mathcal{Z}(B^{(1)}) = \{R \in \mathcal{B}(k): R = B^{(1)} \times B^{(2)} \text{ for some } B^{(2)} \in \mathcal{B}(2)\} \quad (B^{(1)} \in \mathcal{B}(1)), \]
\[ \mathcal{E}(B^{(1)}) = \mathcal{D}(B^{(1)}) \cap \mathcal{Z}(1). \]

Then
\[
A_{2, m} \ast \varphi_{t}(x) = \sum_{B^{(1)} \in \mathcal{E}(m)} \int \varphi_{t}(x - y) \sum_{R \in \mathcal{D}(m, B^{(1)})} b_{R}(y) \, dy \]
\[ - \sum_{B^{(1)} \in \mathcal{E}(m)} \int \varphi_{t}(x - y) \sum_{R \in \mathcal{E}(B^{(1)})} b_{R}(y) \, dy \]
\[ = H_{m} - I_{m}, \quad \text{say.} \]

As in the estimate for \( A_{1} \ast \varphi_{t}(x) \), we have
\[
(3.11) \quad \left| \sum_{m = -\infty}^{\infty} I_{m} \right| \leq c \left( \frac{|S_{+} \cap U_{k}|}{|S|} \right)^{\delta} \left\{ \sum_{R \in \mathcal{B}(k)} M_{S}(b_{R})(x)^{2} \right\}^{1/2}.
\]

It remains to estimate \( \sum_{m} H_{m} \). Let
\[ G_{B^{(1)}}(y) = \sum_{R \in \mathcal{D}(B^{(1)})} b_{R}(y) \quad (B^{(1)} \in \mathcal{B}(1)). \]

Then, for \( B^{(1)} \in \mathcal{E}(m) \), consider the integral:
\[ J = \int \varphi_{t}(x^{(1)} - y^{(1)}, x^{(2)} - y^{(2)}) G_{B^{(1)}}(y^{(1)}, y^{(2)}) \, dy^{(1)}. \]

If we fix \( y^{(2)} \) and regard \( G_{B^{(1)}} \) as a function of \( y^{(1)} \), then \( G_{B^{(1)}} \) has vanishing moments up to the order \( N_{1} - 1 \) and is supported in \( B^{(1)}_{+} \). By Taylor's formula, this implies that
\[
(3.12) \quad |J| \leq c t_{1}^{-N_{1}} t_{2}^{-N_{1}} \left( \frac{2m}{t_{1}} \right)^{N_{1}} \int_{B^{(1)}_{+}} |G_{B^{(1)}}(y^{(1)}, y^{(2)})| \, dy^{(1)}. \]

Note that if \( B^{(1)} \in \mathcal{E}(m) \), then \( B^{(1)}_{+} \subset B^{(1)}(x^{(1)}, 20t_{1}) \). Thus, integrating \( J \) with respect to \( y^{(2)} \) and using (3.12), we find
\[
\left| \int \varphi_{t}(x - y) G_{B^{(1)}}(y) \, dy \right| \leq c \left( \frac{2m}{t_{1}} \right)^{N_{1}} |S|^{-1} \int_{S_{+}} |G_{B^{(1)}}(y)| \, dy.
\]

Consequently
\[
(3.13) \quad |H_{m}| \leq c \left( \frac{2m}{t_{1}} \right)^{N_{1}} |S|^{-1} \int_{S_{+}} \sum_{B^{(1)} \in \mathcal{E}(m)} |G_{B^{(1)}}(y)| \, dy.
\]
Let $F_m(y) = \sum_{B^{(1)} \in \mathscr{B}(1,m)} |G_{B^{(1)}}(y)|^2$. Since by Remark (3.4) we have

$$\sum_{B^{(1)} \in \mathscr{B}(1,m)} \chi_{B^{(1)}} \leq c$$

(where $c$ is independent of $m$), from the Schwarz inequality it follows that

$$\sum_{B^{(1)} \in \mathscr{B}(m)} |G_{B^{(1)}}| \leq cF_m^{1/2}.$$ 

Therefore, by (3.13) we have

$$|H_m| \leq c(2^m/t_1)^{N_1} M_S(F_m^{1/2}),$$

so that

$$\left| \sum_m H_m \right| \leq c \sum_m \left( \frac{2^m}{t_1} \right)^{N_1} M_S(F_m^{1/2}),$$

where the summation with respect to $m$ is taken over all integers $m$ such that $\mathscr{M}(m)$ is not empty. Note that if $\mathscr{M}(m)$ is not empty,

$$(2^m/t_1)^{1/2} \leq c|S_+ \cap U_k|/|S|.$$ 

Thus, applying the Schwarz inequality to the right-hand side of (3.14), we have

$$\left( \sum_m H_m \right)^{1/2} \leq c \left( \frac{|S_+ \cap U_k|}{|S|} \right)^{N_1/2} \left\{ \sum_{m=-\infty}^{\infty} M_S(F_m^{1/2})^2 \right\}^{1/2}.$$ 

On the other hand, by (3.6) and (3.7)

$$\int \sum_{m=-\infty}^{\infty} M_S(F_m^{1/2})^2 \, dx \leq c \sum_{m=-\infty}^{\infty} \int F_m \, dx \leq c \sum_{m=-\infty}^{\infty} \sum_{B^{(1)} \in \mathscr{B}(1,m)} R_{(B^{(1)})}^2$$

$$\leq c2^{2k}|O_k|.$$ 

Therefore, combining (3.15) with (3.11), we obtain a desired estimate. $A_3 \ast \varphi_\delta(x)$ can be treated similarly.

Finally we estimate $A_4 \ast \varphi_\delta(x)$. Note that if $\mathscr{L}(4)$ is not empty, then $U_k \supset S_+$. Thus

$$|A_4 \ast \varphi_\delta(x)| \leq \sum_{j=1}^{3} |A_j \ast \varphi_\delta(x)| + cM_S(a_k)(x)$$

$$\leq \sum_{j=1}^{3} |A_j \ast \varphi_\delta(x)| + c \left( \frac{|S_+ \cap U_k|}{|S|} \right)^\delta M_S(a_k).$$

This is what we need since we have already obtained desired estimates for $A_j \ast \varphi_\delta(x)$ ($j = 1, 2, 3$) and by (3.6) and (3.7) we have $||a_k||_2^2 \leq c2^{2k}|O_k|$. This completes a proof of (3.8).

4. Double singular integrals. Let $K^{(i)} \in C^\infty(\mathbb{R}^n_+ - \{0\})$ be such that

$$\int_{|x^{(i)}|=1} K^{(i)}(x^{(i)})(P_1x^{(i)}, x^{(i)}) \, d\sigma(x^{(i)}) = 0,$$

$$K^{(i)}(A_{t_i}^{(i)}x^{(i)}) = t_i^{-n} K^{(i)}(x^{(i)}) \quad \text{for all } t_i > 0,$$

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where \( d\sigma(x^{(i)}) \) is the area element of \( S^{n_i-1} = \{ x^{(i)} : |x^{(i)}| = 1 \} \). For \( \varepsilon_1, \varepsilon_2 > 0 \), set
\[
K_{\varepsilon_1, \varepsilon_2}(x) = \prod_{i=1,2} K^{(i)}(x^{(i)}) \{ 1 - \chi_{[0,1]}(\varepsilon_1^{-1} \rho^{(i)}(x^{(i)})) \}.
\]
We can generalize the weak type estimates of [3 and 7].

**Theorem 3.** Let \( A \) and \( B \) be compact sets of \( \mathbb{R}^n \). If \( f \) is a function on \( \mathbb{R}^n \) such that \( \int_B |f| \log(2 + |f|) \, dx < \infty \) and \( \text{supp } f \subset B \), then
\[
\left\{ x \in A : \sup_{\varepsilon_1, \varepsilon_2 > 0} |f \ast K_{\varepsilon_1, \varepsilon_2}(x)| > 1 \right\} \leq c \int_B |f| \log(2 + |f|) \, dx,
\]
where \( c \) is a constant independent of \( f \).

Combining results of [7] with Theorem 2, we obtain the equivalence with respect to the \( L^p \)-norms of Lusin functions and radial maximal functions (radial maximal functions can be replaced by nontangential maximal functions). Now we briefly see how this equivalence implies Theorem 3. Let \( h \in C^\infty(\mathbb{R}^1) \) be a nonnegative function such that \( h(u) = 1 \) if \( u < 1 \) and \( h(u) = 0 \) if \( u \geq 2 \). Set
\[
K'_{\varepsilon_1, \varepsilon_2}(x) = K_{\varepsilon_1, \varepsilon_2}(x) \prod_{i=1,2} h(M \rho^{(i)}(x^{(i)}))
\]
where \( M > 0 \). Then, by Stein’s theorem on limits of sequences of operators, to prove Theorem 3 it is sufficient to show that
\[
\sup_{\varepsilon_1, \varepsilon_2 > 0} |f \ast K'_{\varepsilon_1, \varepsilon_2}(x)| < \infty \quad \text{for almost every } x,
\]
where \( f \) is a function with compact support such that \( \int |f| \log(2 + |f|) \, dx < \infty \), \( \int f(x^{(1)}, x^{(2)}) \, dx^{(2)} = 0 \) for all \( x^{(1)} \in \mathbb{R}^{n_1}, \int f(x^{(1)}, x^{(2)}) \, dx^{(1)} = 0 \) for all \( x^{(2)} \in \mathbb{R}^{n_2} \). We can show by a direct estimate that \( f^+ \in L^p \) for some \( p < 1 \). Thus, as in [7], (4.1) follows from the inequality
\[
\sup_{\delta > 0} \| (f \ast K^{(\delta)})^+ \|_p \leq c \| f^+ \|_p,
\]
where
\[
K^{(\delta)}(x) = \prod_{i=1,2} K^{(i)}(x^{(i)}) h(M \rho^{(i)}(x^{(i)})) \{ 1 - h(\delta^{-1} \rho^{(i)}(x^{(i)})) \}.
\]
(4.2) is proved as follows. First, by Theorem 2 we have that \( \| (f \ast K^{(\delta)})^+ \|_p \leq c \| S(f \ast K^{(\delta)}) \|_p \). Next, it is not difficult to see that \( \sup_{\delta > 0} \| S(f \ast K^{(\delta)}) \|_p \leq c \| S(f) \|_p \). Finally, from results of [7] it follows that \( \| S(f) \|_p \leq c \| f^+ \|_p \). Combining these results, we obtain (4.2). See [7] for more details.

**References**


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