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DIMENSION OF THE GRAPH OF RIEMANN INTEGRABLE FUNCTIONS

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ABSTRACT. The Hausdorff $h-m$ measure of the graph of a Riemann integrable function is shown to be finite provided h satisfies an inequality related to the rate of convergence of the upper and lower Riemann sums.

Besicovitch and Ursell showed in [**1**] that a function which satisfies a Lipschitz condition of order δ has a graph whose Hausdorff dimension d (cf. [**3**, p. 149 ff.] for generalizations) satisfies $1 \le d \le 2-\delta$. Mauldin and Williams [**2**] extended this result to continuous functions which satisfy a condition they call convex Lipschitz of order θ. These results form the motivation for the method of estimating the dimension of graphs of Riemann integrable functions given below. The rate at which the upper and lower Riemann sums converge will be used to determine a cover of the graph which will provide an estimate of the Hausdorff dimension.

We consider a nonconstant $f(x)$ defined on $[a,b]$. If $f(x)$ is Riemann integrable, then for each $\varepsilon > 0$ there is a $\delta > 0$ so that

$$(*) \qquad \begin{array}{l}\text{whenever } a = x_0 < x_1 < \cdots < x_n = b,\ x_i - x_{i-1} < \delta \text{ and} \\ z_i,\ w_i \in [x_{i-1}, x_i] \text{ it follows that } \sum (f(z_i) - f(w_i))\Delta x_i < \varepsilon.\end{array}$$

Thus $\varepsilon > 0$ determines a function $\delta(\varepsilon)$ equal to the supremum of the set of δ for which $(*)$ holds. Since the function $\delta(\varepsilon)$ is nondecreasing and $\delta(0+) = 0$, it determines a function $\varepsilon(t)$ which is continuous on the right at each t in the range of $\delta(\varepsilon)$ and satisfies $\delta(\varepsilon(t)) = t$.

Let $h(t)$ be a nondecreasing function, continuous on the right at each t and satisfying $h(0+) = 0$. Then h determines a Hausdorff $h-m$ measure (cf. [**3**]). We consider only those functions h which satisfy the condition that $h(t)/t$ is bounded. These are the functions which would be normally employed for sets whose dimension is greater than or equal to 1.

THEOREM. *Given h and f as above and the function $\varepsilon(t)$ determined by the Riemann integrability of f, suppose that there is a sequence $\delta_n \downarrow 0$ and a number M such that $h(\delta_n\sqrt{2}) \le M\delta_n^2/\varepsilon(\delta_n)$. Then the $h-m$ measure of the graph of f is finite.*

PROOF. Let $\delta_n \downarrow 0$ as in the statement of the theorem. Fix n and let $\delta = \delta_n$. Let $k = [(b-a)/\delta]+1$ where $[x]$ is the greatest integer less than x. Let $x_i = a + i\cdot\delta$, $i = 0, 1, \ldots, k-1$, and let $x_k = b$. For $i = 1, 2, \ldots, k$ let $n_i = [\sup(f(w_i) - f(z_i))/\delta]$ where the supremum is taken over all $w_i,\ z_i \in [x_{i-1}, x_i]$. Then the graph of f

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on $[a,b]$ can be covered with $n_1+n_2+\cdots+n_k+k$ squares of side length δ and $(n_1+n_2+\cdots+n_{k-1})\delta^2<\varepsilon(\delta)$. Thus

$$(n_1+n_2+\cdots+n_{k-1})h(\delta\sqrt{2})\le\varepsilon(\delta)h(\delta\sqrt{2})/\delta^2\le M.$$

The remaining n_k+k squares covering the graph of f are handled as follows: Since f is Riemann integrable, there is a number N_1 so that $|f(x)|\le N_1$. There is also an N_2 so that $h(\delta)/\delta\le N_2$. It follows that

$$kh(\delta\sqrt{2})\le\frac{b-a+\delta}{\delta}h(\delta\sqrt{2})<(b-a+\delta)2N_2$$

and because $n_k\delta\le 2N_1$ it follows that

$$n_kh(\delta\sqrt{2})\le\frac{2N_1}{\delta}h(\delta\sqrt{2})\le\sqrt{2}N_1N_2.$$

Because of the above, the Hausdorff $h-m$ measure of the graph of f is less than $M+2(b-a)N_2+\sqrt{2}N_1N_2$.

Suppose that f satisfies a Lipschitz condition of order α; i.e., there exists m such that $|f(x+\Delta x)-f(x)|\le m|\Delta x|^\alpha$. Let $(b-a)/n=\delta=\Delta x_i$. Then

$$\sum_{i=1}^n(f(w_i)-f(z_i))\Delta x_i\le\sum_{i=1}^n m\Delta x_i^\alpha\Delta x_i\le m\cdot n\cdot\delta^{\alpha+1}=m(b-a)\delta^\alpha.$$

Thus $\varepsilon(\delta)\le m(b-a)\delta^\alpha$. But

$$h(\delta\sqrt{2})=\sqrt{2}^{2-\alpha}\delta^{2-\alpha}=\frac{m(b-a)\sqrt{2}^{2-\alpha}\delta^2}{m(b-a)\delta^\alpha}\le\frac{m(b-a)\sqrt{2}^{2-\alpha}\delta^2}{\varepsilon(\delta)}$$

and the hypotheses of the theorem are satisfied. Thus, the Besicovitch and Ursell result for Lipschitz functions follows from the theorem. One can also note that Part III in [**1**] can be obtained by applying the above theorem. In addition, the theorem can indicate a bound on the dimension of the graph of a function when the function does not satisfy a Lipschitz condition or when the function is not continuous. Easily checked examples of this are: $f(x)=x^{1/2}$ on $[0,1]$ or $g(x)=0$ if $x\notin C$, $g(x)$ bounded on $C=$ the Cantor ternary set.

Finally we note that the result can be generalized to the graphs of real valued functions of several variables. Suppose that $f\colon X\to R$ where X is a rectangle in Euclidean n-space and f is Riemann integrable. Suppose that $h(t)/t^n$ is bounded and $\delta(\varepsilon)$ and $\varepsilon(t)$ are defined as in the above. Then, if there exists $\delta_n\downarrow 0$ and a number M such that

$$h(\delta_n/\sqrt{n+1})\le M\cdot\delta_n^{n+1}/\varepsilon(\delta_n),$$

the $h-m$ measure of the graph of f is finite.

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