THE DERIVATIVE OF BAZILEVIČ FUNCTIONS

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ABSTRACT. For \( \alpha > 0 \), let \( B_1(\alpha) \) be the class of normalised analytic functions \( f \) defined in the open unit disc \( D \) such that \( \text{Re}(f(z)/z)^{\alpha-1}f'(z) > 0 \) for \( z \in D \). Sharp upper and lower bounds are obtained for \( |zf'(z)/f(z)| \) when \( f \in B_1(\alpha) \).

1. Introduction. For \( \alpha > 0 \), denote by \( B(\alpha) \) the class of analytic Bazilevic functions defined in the unit disc \( D \), with \( f(0) = 0 \) and \( f'(0) = 1 \) (e.g. \([2, 8]\)) and by \( B_1(\alpha) \) the subclass of \( B(\alpha) \) for which

\[
\text{Re} f'(z)(f(z)/z)^{\alpha-1} > 0
\]

for \( z \in D \) \([7]\). Clearly \( B_1(1) = R \), the class of analytic functions satisfying \( \text{Re} f'(z) > 0 \) in \( D \) first studied by Alexander \([1]\).

In \([9]\), it was shown that for \( f \in R \) and \( z \in D \),

\[
\left| \frac{zf'(z)}{f(z)} \right| \leq \frac{-K}{(1-|z|)\log(1-|z|)},
\]

where \( K \) is an absolute constant. Recently, London \([5]\) obtained the sharp upper bound and Gray and Ruscheweyh \([4]\), the sharp upper and lower bounds for \( |zf'(z)/f(z)| \) when \( f \in R \).

In this paper, we give sharp upper and lower bounds for the wider class \( B_1(\alpha) \). This sharpens the upper bound estimate given by El-Ashwah and Thomas \([3]\).

2. Results. Following Gray and Ruscheweyh (loc. cit), we begin by defining a slightly wider class of functions.

DEFINITION. For \( \alpha > 0 \), denote by \( B_0(\alpha) \) the class of function analytic in \( D \) with \( f(0) = 0 \), \( f'(0) = 1 \) and satisfying the condition

\[
\text{Re} e^{i\phi} f'(z)(f(z)/z)^{\alpha-1} > 0
\]

for \( z \in D \) and for some \( \phi = \phi(f) \in \mathbb{R} \).

THEOREM. For \( f \in B_0(\alpha) \) and \( |z| \leq r < 1 \),

\[
\frac{1-r}{\alpha(1+r)} \int_0^1 t^{\alpha-1} \frac{1-tr}{1+tr} dt \leq \left| \frac{zf'(z)}{f(z)} \right| \leq \frac{1+r}{\alpha(1-r)} \int_0^1 t^{\alpha-1} \frac{1+tr}{1-tr} dt.
\]

The left-hand and right-hand inequalities are sharp in \( B_0(\alpha) \) for the function

\[
f_0(z) = z \left( \alpha \int_0^1 t^{\alpha-1} \frac{1+tz}{1-tz} dt \right)^{1/\alpha}
\]

at \( z = -r \) and \( z = r \) respectively.

We use the method of Gray and Ruscheweyh (loc. cit) and require the following lemma.

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LEMMA. Let $F(z) = 1 - z^a / (\alpha \int_0^z \zeta^{\alpha-1} (1 - \zeta) \, d\zeta)$ and $G(z) = (1 - F(z)) / (1 - z)$. Then $F$ and $G$ have nonnegative Taylor coefficients about $z = 0$, and in particular for $|z| < r < 1$,

$$|F(z)| < F(r) < \lim_{t \to 1} F(t) = 1,$$

and

$$|F'(z)| < F'(r)$$

and

$$|G(z)| < G(r).$$

Proof. Let

$$H(z) = F(z) - 1 = -z^a / \left( \alpha \int_0^z \zeta^{\alpha-1} \frac{1}{1 - \zeta} \, d\zeta \right).$$

Then clearly

$$(1 - z)(zH'(z) - \alpha H(z)) = \alpha H^2(z).$$

With $H(z) = \sum_{k=0}^{\infty} c_k z^k$, (5) implies that

$$(k - \alpha)c_k = (k - 1 - \alpha)c_{k-1} + \alpha \sum_{j=0}^{k} c_j c_{k-j},$$

where $c_{-1} = 0$. Thus

$$c_0 = -1, \quad c_1 = \frac{\alpha}{\alpha + 1}, \quad c_2 = \frac{\alpha}{(2 + \alpha)(\alpha + 1)^2}$$

and for $k \geq 3$,

$$(k + \alpha)c_k = \left( k + \frac{\alpha^2 - 2\alpha - 1}{\alpha + 1} \right) c_{k-1} + b_k,$$

where

$$b_3 = 0 \text{ and } b_k = \alpha \sum_{j=2}^{k-2} c_j c_{k-j} \quad \text{for } k \geq 4.$$

Since $3 + (\alpha^2 - 2\alpha - 1)/(\alpha + 1) > 0$ a simple induction argument using (6) and (7) shows that $c_k > 0$ for $k \geq 1$. Thus the coefficients of $F$ are nonnegative and (2) and (3) follow. Finally, with $G(z) = \sum_{k=0}^{\infty} d_k z^k$, we have

$$d_k = 1 - \sum_{j=1}^{k} c_j = 1 - \lim_{t \to 1} \sum_{j=1}^{k} c_j t^j \geq 1 - \lim_{t \to 1} F(t) = 0$$

and (4) follows.

Proof of the Theorem. From (1), it follows that

$$\frac{zf'(z)}{f(z)} = \frac{h(z)}{\alpha z^{-\alpha} \int_0^z \zeta^{\alpha-1} h(\zeta) \, d\zeta} = \frac{h(z)}{\alpha \int_0^1 t^{\alpha-1} h(tz) \, dt}$$

where $\text{Re} e^{i\phi} h(z) > 0$ for $z \in D$. It follows from the Duality Principle [6, Theorem 1.1, Corollary 1.1 and Theorem 1.6] that any value assumed by the right-hand side of (8) for some $z \in D$ is also assumed for this $z$ when $h$ is a function of the form
The derivative of Bazilević functions

\[ \frac{(1 + xz)/(1 + yz)}{f(z)} \] where \(|x|, |y| = 1\). Clearly in obtaining upper and lower bounds for \(|zf'(z)/f(z)|\), we may take

\[ h(z) = \frac{1 + xz}{1 - z} \text{ for } |z| = 1. \]

We first obtain the lower bound in the Theorem. Using (8) and (9), we write

\[
\frac{f(z)}{zf'(z)} = \frac{1 - z}{1 + xz} \int_0^z \frac{\zeta^\alpha - 1}{1 - \zeta} \cdot \frac{1 + xz}{1 - tz} d\zeta.
\]

Now for \(0 < t < 1\) and \(|z| < 1\),

\[
\frac{1 + tz}{1 + |z|} \leq \frac{1 - t|z|}{1 - |z|}.
\]

Thus

\[
\frac{1 + xtz}{1 + xz} \frac{1 - z}{1 - tz} \leq \frac{1 - t|z|}{1 - |z|} \frac{1 + xz}{1 + t|z|}
\]

and so

\[
\left| \frac{f(z)}{zf'(z)} \right| \leq \alpha \frac{1 + r}{1 - r} \int_0^1 t^{\alpha - 1} \frac{1 - tr}{1 + tr} dt
\]

which is the required lower bound.

For the upper bound, we use (9) together with \(F\) as defined in the Lemma to write

\[
\alpha \int_0^z \zeta^{\alpha - 1} h(\zeta) d\zeta = \alpha \int_0^z \zeta^{\alpha - 1} \left( -z + \frac{z + 1}{1 - \zeta} \right) d\zeta = z^\alpha \frac{1 + xF(z)}{1 - F(z)}.
\]

Hence (8) and (9) give

\[
\frac{zf'(z)}{f(z)} = G(z) \frac{1 + xz}{1 + xF(z)},
\]

where \(G(z) = (1 - F(z))/(1 - z)\). Since \((1 + az)/(1 + b)\) maps the closed unit disc onto the circle centre \((1 - ab)/(1 - b^2)\), radius \(|a - b|/(1 - b^2)|\) provided \(|b| < 1\), we deduce that

\[
\left| \frac{zf'(z)}{f(z)} \right| \leq |G(z)| \frac{|z - F(z)| + |1 - F(z)|^2}{1 - |F(z)|^2} \leq \frac{|G(z)|}{1 - |F(z)|^2} \left( r \left| 1 - \frac{F(z)}{z} \right| + 1 - r^2 + r^2 \left( 1 - \frac{F(z)}{z} \right)^2 \right) \leq \frac{1 + r}{1 - |F(z)|^2} \left( \frac{r}{\alpha} |F'(z)| + (1 - r)|G(z)| \right)
\]

where we have used \(F'(z) = \alpha G(z)(1 - F(z))/z\).
It now follows from the Lemma that the last expression is maximal for $z = r$ and so
\[
\frac{|zf'(z)|}{f(z)} = \frac{(1 + r)G(r)}{1 + F(r)} = \frac{1 + r}{1 - r} \frac{1 - F(r)}{1 + F(r)}
\]
\[
= \frac{1 + r}{1 - r} \left( -1 + 2\alpha r^{-\alpha} \int_0^r \frac{\zeta^{\alpha-1}}{1 - \zeta} d\zeta \right)^{-1}
\]
\[
= (1 + r) \left( \alpha(1 - r) \int_0^1 t^{\alpha-1} \frac{1 + tr}{1 - tr} dt \right)^{-1}
\]
which completes the proof.

REFERENCES

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