ABSTRACT. It is shown that the C*-algebra $M(A)/A$, where $A$ is a nonunital separable simple AF C*-algebra and $M(A)$ is the multiplier algebra of $A$, is simple if and only if $A$ has a continuous scale or $A$ is elementary. Some results concerning the ideal structure of $M(A)/A$ are also obtained in the case that it is nontrivial.

1. Introduction. Let $K$ denote the C*-algebra of all compact operators on a separable Hilbert space $H$, and $B(H)$ the C*-algebra of all bounded operators on $H$. Then $B(H)$ is the multiplier algebra of $K$. (The multiplier algebra of a C*-algebra is the idealiser of the C*-algebra in its double dual.) It is well known that $B(H)/K$ is simple. Let $A$ be a separable simple AF C*-algebra with multiplier algebra $M(A)$. When is $M(A)/A$ simple? Elliott showed [4] that if $A$ is an infinite, nonelementary separable matroid C*-algebra (which is a simple AF C*-algebra) then $M(A)/A$ has precisely one nonzero proper (closed, two sided) ideal. He also showed that if $A$ is a finite separable matroid C*-algebra, then $M(A)/A$ is simple. In this paper we shall consider a separable simple AF C*-algebra $A$. We shall show that $M(A)/A$ is simple if and only if either $A$ has a continuous scale or $A = K$. We shall also give some other results concerning the ideal structure of $M(A)/A$.

Recall that a separable C*-algebra $A$ is AF if whenever $a_1, \ldots, a_n \in A$ and $\varepsilon > 0$ are given, there exist a finite dimensional C*-subalgebra $B$ of $A$ and elements $b_1, \ldots, b_n \in B$ such that $\|a_i - b_i\| < \varepsilon$, $i = 1, 2, \ldots, n$. Furthermore, if we are initially also given a finite dimension C*-subalgebra $B_0$, we may choose $B \supseteq B_0$.

Let $A$ be a nonelementary separable simple AF C*-algebra and $G$ the corresponding simple dimension group with scale $\Gamma(G)$. Fix an element $u \in G^+ \backslash \{0\}$. Let $\mathcal{S} = S_u(G)$ denote the set of all homomorphisms $\tau : G \to \mathbb{R}$ such that $\tau(G^+) \geq 0$ and $\tau(u) = 1$. Then $\mathcal{S}$ is a convex compact subset of the locally convex space $\mathbb{R}^G$ of all functions $f : G \to \mathbb{R}$ with the product topology. Each $\tau \in \mathcal{S}$ can be viewed as a trace on $A$ such that for each projection $p \in A$, $\tau(p) < \infty$. We shall denote the extreme points of $\mathcal{S}$ by $E(\mathcal{S})$. Let $\text{Aff}(\mathcal{S})$ denote the set of all affine, real continuous functions on $\mathcal{S}$. We have a positive homomorphism $\theta : G \to \text{Aff}(\mathcal{S})$, $a \to \hat{a}$, where $\hat{a}(\tau) = \tau(a)$. By [3, Corollary 4.2], $\theta$ determines the order on $G$ in the sense that $G^+ = \{a \in G : \hat{a} \geq 0\} \cup \{0\}$. Hence $G^+ \cap \ker \theta = \{0\}$. Moreover, $S = S_u(G)$ is a Choquet simplex and $H = \theta(G)$ is a dense additive subgroup of $\text{Aff}(\mathcal{S})$. For the details of simple dimension groups readers are referred to [3, Chapter 4].
For every $\tau \in S$ (as a trace), we can extend $\tau$ to a trace on $M(A)_+$. In particular, $\tau(1) = \sup\{\tau(e_n): n = 1, 2, \ldots\}$, where $\{e_n\}$ is an approximate identity for $A$ consisting of projections, and 1 is the unit of $M(A)$. As in [5, Theorem 2], one can easily show that $\Gamma(G) = \{a \in G^+: \tilde{a}(\tau) < \tau(1) \text{ for all } \tau \in S\}$, provided that $A$ is nonunital.

We say that $A$ has a continuous scale if $\hat{\tau}(\tau)$ is bounded and continuous on $S$, and a bounded scale if $\hat{\tau}(\tau)$ is bounded on $S$; we say that $A$ is finite if $\hat{\tau}(\tau) < \infty$ for all $\tau \in S$. We say that $A$ is infinite if $A$ is not finite, and that $A$ is stable if $\hat{\tau}(\tau) \to \infty$ for all $\tau \in S$, which is equivalent to saying that $A \cong A \otimes K$ (see [2, Theorem 4.9]).

2. Simplicity of $M(A)/A$.

**Lemma 1.** Let $A$ be a nonelementary separable infinite simple AF $C^*$-algebra. Let $F = \{\tau \in S: \hat{\tau}(\tau) = \infty\}$ and let $J_\tau$ be the closure of the set $\{a \in M(A): \tau(a^{*}a) < \infty\}$ where $\tau$ is a fixed element in $F$. Then $J_\tau$ is an ideal of $M(A)$ such that $A \subseteq J_\tau \subset M(A)$.

**Proof.** Let $J_\tau^0 = \{a \in M(A): \tau(a^{*}a) < \infty\}$. Then $J_\tau^0$ is a $\tau$-invariant linear subspace of $M(A)$. Let $a \in J_\tau^0, b \in M(A)$. Then

$$|\tau(a^{*}b^{*}ba)| \leq ||b||^2 \tau(a^{*}a) < \infty.$$ 

Hence $ba \in J_\tau^0$; similarly $ab \in J_\tau^0$. Thus $J_\tau$ is a closed ideal of $M(A)$. Since for every projection $p \in A$, $\tau(p) < \infty$ for all $\tau \in S$, and $A$ is AF, we conclude that $A \subseteq J_\tau$. Let $\{e_n\}$ be an approximate identity of $A$ consisting of projections, and set $f_n = e_n - e_{n-1}$ ($e_0 = 0$). Since $\theta(G)$ is dense in Aff($S$), there are projections $q_n \in A$ such that $0 \leq \theta[q_n] \leq 2^{-n} (1/\theta[f_n]) \theta[f_n]$, if $\theta[f_n] > 1$, or $0 < \theta[q_n] \leq 2^{-n} \theta[f_n]$, if $\theta[f_n] \leq 1$, where $||\theta[f_n]|| = \sup\{\tau(f_n): \tau \in S\}$. We may assume that $q_n \leq f_n$. Since $q_n \neq 0$ and $q_n \leq f_n$, we have that $q = \sum_{n=1}^{\infty} q_n$ is a projection in $M(A)$ but not in $A$. Moreover $\tau(q) = \sum_{n=1}^{\infty} \tau(q_n) \leq 1 < \infty$. So $q \in J_\tau^0 \subseteq J_\tau$. Hence $J_\tau \supseteq A$.

Now we show that $1 \notin J_\tau$. Otherwise there is $a \in (J_\tau^0)_+$ such that $||1 - a|| < \frac{1}{4}$. Thus $sp(a) \subset (\frac{3}{4}, \frac{5}{4})$. This implies that $0 \leq 1 \leq \frac{3}{4} a$. Then $\tau(1) < \infty$, a contradiction.

Blackadar showed in [2, Theorem 4.8] that $A$ has a bounded scale if, and only if, $A$ is algebraically simple. If $A$ has a bounded scale, then must $M(A)/A$ be simple?

We will see after the following lemma.

**Lemma 2.** Let $A$ be a nonunital, nonelementary simple AF $C^*$-algebra. Let $I_0$ be the closure of

$$I_{00} = \{a \in M(A): \text{ there is } \{a_n\} \subset A \text{ such that } \tau((a - a_n)^{*}(a - a_n)) \text{ converges to zero uniformly on } S\}.$$ 

Then

1. $I_0$ is a (closed) ideal of $M(A)$, $A \not\subseteq I_0 \subseteq M(A)$, and $I_0$ is the smallest such ideal.
2. If $A$ is algebraically simple, then $I_{00}$ is already closed.
3. If $A$ has no continuous scale, $I_0 \not\subseteq M(A)$.

**Proof.** Clearly $I_{00}$ is a $\tau$-invariant linear subspace of $M(A)$ containing $A$. Suppose that $a \in M(A), b \in M(A)$, and $a_n \in A$ are such that $\tau((a-a_n)^{*}(a-a_n)) \to 0$ uniformly on $S$. We have

$$\tau\left[\tau[(b(a-a_n)^{*})(b'(a-a_n))]\right] \leq ||b||^2 \tau((a-a_n)^{*}(a-a_n)) \to 0$$
uniformly on $S$. Since $ba_n \in A$, by the definition of $I_{00}$, $ba \in I_{00}$. Similarly $ab \in I_{00}$. Hence $I_{00}$ is an ideal of $M(A)$. So $I_0$ is a closed ideal of $M(A)$. The projection $q$ constructed in the first part of the proof of Lemma 1 is continuous on $S$. Moreover, we see that $\tau\left(\sum_{k=1}^{n} q_k\right)$ converges uniformly to $\tau(q)$ on $S$. Hence

$$\tau\left(\left(q - \sum_{k=1}^{n} q_k\right)^*\left(q - \sum_{k=1}^{n} q_k\right)\right) = \tau\left(q - \sum_{k=1}^{n} q_k\right) \to 0$$

uniformly on $S$. Thus $q \in I_0 \setminus A$ and $I_0 \not\subset A$.

Let $g$ be a projection in $I_0$, and let us show that $g \in I_{00}$. Since $I_{00}$ is dense in $I_0$, we have that $gI_{00}g$ contains a positive element close to $g$, and hence contains $g$.

Suppose that $I$ is another ideal such that $I \not\subset A$. Let $\{e_k\}$ be an approximate identity of $A$ consisting of projections, and set $f_n = e_n - e_{n-1}$ ($e_0 = 0$). As in the proof of [4, Theorem 3.1], there is a projection $p \in I \setminus A$ such that $e_kp = pe_k$. To show $I_0 \subseteq I$, it is enough to show that every projection $g \in I_0$ satisfying $e_kg = ge_k$ is in $I$, as in the proof of [4, Theorem 3.2]. Let $g$ be such a projection. Then $g = \sum f_n$. Also, $g \in I_{00}$, and so $\tau(g)$ is finite and continuous on $S$; therefore $\sum_{k=1}^{n} \tau(gf_k)$ converges to $\tau(g)$ uniformly on $S$, by Dini’s theorem. We may assume that $p_{f_1} \neq 0$; then $\{\tau(pf_k) : \tau \in S\} > 0$. Since $\sum_{n=1}^{\infty} \tau(gf_n)$ converges uniformly on $S$, we can choose an integer $n_0$ such that

$$\sum_{k \geq n_0} \tau(gf_k) < \tau(pf_1) \quad \text{for all} \quad \tau \in S.$$ 

Then since infinitely many $pf_n$ are nonzero, there exists a partition of the set $\{n_0 + 1, n_0 + 2, \ldots\}$ into finite subsets $N_1, N_2, \ldots$ (of consecutive integers) such that for each $n = 1, 2, \ldots$, either $N_n = \emptyset$ or

$$\sum_{k \in N_n} \tau(gf_k) < \tau(pf_n) \quad \text{for all} \quad \tau \in S.$$ 

Thus $\left|\sum_{k \in N_n} gf_k\right| < |pf_n|$. There exists for each $n = 1, 2, \ldots, u_n \in A$ such that $u_nu_n^* = \sum_{k \in N_n} gf_k$ and $u_nu_n \leq pf_n$. Set $u = \sum_{n=1}^{\infty} u_n$. Then $u \in M(A)$, $uu^* = g - ge_{n_0}$, and $up = u$. Hence $u, u^*$, and $g$ are in $I$. So $I_0 \subseteq I$.

Now suppose that $A$ is algebraically simple, and let $a \in M(A)$, $b_n \in I_{00}$ be such that $\|b_n - a\| \to 0$.

We may assume that $|a - b_n| \leq 1$. Then

$$\left|\tau((a - b_n)^*(a - b_n))\right| \leq \|a - b_n\|\tau(|a - b_n|).$$

Since $A$ has a bounded scale, $\tau(|a - b_n|) \leq \tau(1) \leq N$, for all $\tau \in S$ and some $N > 0$. Hence $\tau((a - b_n)^*(a - b_n)) \to 0$ uniformly on $S$. Let $a_n \in A$ be such that $\tau((b_n - a_n)^*(b_n - a_n)) < 1/n$ uniformly on $S$. We have

$$\tau((a - a_n)^*(a - a_n))^{1/2} \leq \tau((a - b_n)^*(a - b_n))^{1/2} + \tau((b_n - a_n)^*(b_n - a_n))^{1/2} \to 0$$

uniformly on $S$. We conclude that $I_{00}$ is closed.

Finally suppose that $A$ has no continuous scale. Then $1 \notin I_0$, i.e. $I_0 \not\subset M(A)$.

**THEOREM 1.** Let $A$ be a separable simple AF $C^*$-algebra. Then $M(A)/A$ is simple if, and only if, either $A$ has a continuous scale or $A$ is elementary.

**Proof.** Suppose that $A$ is not elementary and has no continuous scale. By Lemma 2, $I_0$ is a closed ideal of $M(A)$ such that $A \not\subset I_0 \not\subset M(A)$. In other words, $M(A)/A$ is not simple.
If $A$ is elementary, it is well known that $M(A)/A$ is simple. We may now assume that $A$ has a continuous scale, i.e. that $\tau(1)$ is finite and continuous on $S$. By Dini’s theorem, $\tau(e_n)$ converges to $\tau(1)$ uniformly on $S$. By the definition of $I_0$, $1 \in I_0$. Hence $I_0 = M(A)$. By Lemma 2, $I_0$ is the smallest ideal containing $A$. We conclude that $M(A)/A$ is simple.

REMARKS. Theorem 1 implies Theorem 3.1 of [4].

Given a simple dimension group $G$ we can construct a separable, nonunital, simple $AF$ $C^*$-algebra $A$ with a continuous scale such that the dimension group of $A$ is $G$. So for every separable, nonunital simple $AF$ $C^*$-algebra $A$, there is a separable, nonunital simple $AF$ $C^*$-algebra $B$ such that $A \otimes K \cong B \otimes K$ and $M(B)/B$ is simple.

3. Ideals of $M(A)/A$. Let $A$ be a nonunital, separable, simple $AF$ $C^*$-algebra, and let $G$ and $S = SU(G)$ be as before. Set $F = \{\tau \in S: \tau(1) = \infty\}$ and let $\alpha$ be a subset of $F \cap E(S)$. Let $I_\alpha$ denote the closure of the set $\{\alpha \in M(A): \tau(\alpha^* \alpha) < \infty \text{ for all } \tau \in \alpha\}$. Then, as is easily seen, $I_\alpha$ is an ideal of $M(A)$ containing $A$. The following theorem is a generalization of Theorem 3.2 of [4].

**Theorem 2.** Let $A$ be a nonunital, nonelementary, separable simple $AF$ $C^*$-algebra. Suppose that $E(S)$ has only finitely many points and $F \cap E(S)$ has $n$ points. Then $M(A)/A$ has exactly $2^n - 1$ different proper closed ideals, each of which has the form $I_\alpha/A$.

**Proof.** Suppose that $n = 0$. Since $E(S)$ has finitely many points, $A$ has a continuous scale. In this case, Theorem 2 follows from Theorem 1.

We now suppose that $F \cap E(S) = \{\tau_1, \ldots, \tau_n\}$, $n \geq 1$. As in the proof of Lemma 1, each $I_\alpha$ is a proper closed ideal of $M(A)$ containing $A$ properly. Let us show that if $\alpha, \beta$ are nonempty subsets of $F \cap E(S)$ with $\alpha \neq \beta$, then $I_\alpha \neq I_\beta$. We may assume that $\alpha = \{\tau_1, \ldots, \tau_k\}$ where $k < n$, and that $\tau_{k+1} \in \beta$. For each $n = 1, 2, \ldots$, let $h_n \in \text{Aff}(S)$ be such that

\[ \theta(f_n)(\tau_{k+1}) > h_n(\tau_{k+1}) > \theta(f_n)(\tau_{k+1}), \]

and

\[ 0 < h_n(\tau_i) \leq \min(2^{-n}, \theta(f_n)(\tau_i)), \quad i = 1, 2, \ldots, k. \]

(Since $E(S)$ is finite, the existence of $h_n$ is clear.) Since $\theta(G)$ is dense in $\text{Aff}(S)$, we may assume that $h_n \in \theta(G)$. So we have projections $p_n \in A$ such that $p_n \leq f_n$ and $\tau_i(p_n) \leq 2^{-n}$, $i = 1, 2, \ldots, k$, $\tau_{k+1}(p_n) \geq \frac{1}{2}\tau_{k+1}(f_n)$. Then with $p = \sum p_n$, we have $p \in M(A)$ and $p \notin I_\beta$; this is proved in the same way as $I_\alpha \neq I_\beta$ in Lemma 1. Thus $I_\alpha \neq I_\beta$.

Suppose that $I$ is a closed ideal of $M(A)$. We shall show that $I$ is equal to the smallest $I_\alpha$ which contains it ($\alpha$ could be the empty set).

Let $I_\alpha$ be such an ideal. Write $F \cap (E(S) \setminus \alpha) = \{\tau_1, \tau_2, \ldots, \tau_s\}$, and set $\alpha_1 = \alpha \cup \{\tau_i\}$. Since $I \notin I_{\alpha_1}$, there are projections $g_i \in I \setminus I_{\alpha_1}$ such that $f_k g_i = g_i f_k$ for $i = 1, 2, \ldots, s$ and $k = 1, 2, \ldots$ (see the proof of [4, Theorem 3.2]). Thus $\tau_i(g_i) = \infty$.

Changing $g_i f_k$ into equivalent projections, we may assume that they belong to a common finite dimensional $C^*$-subalgebra of $f_k A f_k$, say $B_k$. Then the range projection $h_k$ of $\sum_{i=1}^s g_i f_k$ exists in $B_k$. Since $B_k$ is a finite dimensional $C^*$-algebra, $\{(\sum_{i=1}^s g_i f_k)^{1/n} \to h_k \}$ in norm. Hence $\sum_{i=1}^s g_i f_k$ has an inverse $b_k$ in the $C^*$-subalgebra $h_k B_k h_k$. Set $h = \sum_{k=1}^\infty h_k$ and $b = \sum_{k=1}^\infty b_k$. Both $h$ and $b$ are
in \( M(A) \). Since \( h = b(\sum_{i=1}^{s} g_i) \), \( h \in I \). Clearly \( \tau(h) \geq \tau(g_i) \), \( i = 1, 2, \ldots, s \). So \( \tau_i(h) = \infty \) for \( i = 1, 2, \ldots, s \).

As in the proof of [4, Theorem 3.27], to show that \( I \supset I_\alpha \), it is enough to show that every projection \( q \in I_\alpha \) such that \( f_k q = q f_k \) is in \( I \). Suppose that \( q \) is such a projection. There exists a partition of \( \{1, 2, \ldots\} \) into finite sets \( N_1, N_2, \ldots \) (of consecutive integers) such that for each \( m = 1, 2, \ldots, s \),

\[
\tau_i(q f_m) < \sum_{k \in N_i} \tau_i(h_k), \quad i = 1, 2, \ldots, s.
\]

Let \( \beta_m \) denote the set of \( \tau \) in \( E(S) \) such that

\[
\tau(q f_m) > \sum_{k \in N_m} \tau(h_k).
\]

Then \( \beta_m \subset \alpha \cup \{E(S) \setminus F\} \). Since \( \theta(G) \) is dense in \( \text{Aff}(S) \), for each \( m \), there is a projection \( q_m < q f_m \) such that

\[
0 < \tau(q f_m - q_m) < \sum_{k \in N_m} \tau(h_k)
\]

for \( \tau \in \beta_m \) and

\[
0 < \tau(q_m) < 1/2^m \quad \text{for} \quad \tau \in E(S) \setminus \beta_m.
\]

Thus \( q_0 = \sum_{m=1}^{\infty} q_m \) is in \( I_0 \), the closure of \( \{a \in M(A) : \tau(a^* a) < \infty \text{ for all } \tau \in E(S)\} \). Set \( q' = q - q_0 \); then

\[
\tau(q' f_m) < \sum_{k \in N_m} \tau(h_k)
\]

for all \( \tau \in E(S) \), hence for all \( \tau \in S \). Therefore there exists for each \( m = 1, 2, \ldots, v_m \in A \) such that \( v_m* v_m = q' f_m \) and \( v_m* v_m \leq \sum_{k \in N_m} h_k \). Set \( v = \sum_{m=1}^{\infty} v_m \); then \( v \in M(A) \), and \( q' f_m v = v \sum_{k \in N_m} h_k = v_m \). In particular \( v \) is a partial isometry, and \( v* v = q' = q - q_0 \) and \( v h = v \). Then \( v, v* \), and therefore \( q - q_0 \) are in \( I \). Since \( E(S) \) is finite, by Lemma 2, \( I_0 \) is the smallest ideal in \( M(A) \) properly containing \( A \). So \( I \supset I_0 \), whence \( q_0 \in I \) and \( q \in I \). This completes the proof.

**Theorem 3.** Let \( A \) be a nonelementary separable infinite simple AF \( C^* \)-algebra. Suppose that \( F \cap E(S) \) is an infinite set. Then \( M(A)/A \) has infinitely many different (closed) ideals.

**Proof.** Let \( \{\tau_i\} \) be a sequence in \( F \cap E(S) \). Let \( F_k = \{\tau_i : i = 1, 2, \ldots, k + 1\} \) and let \( J_k \) be the closure of \( \{a \in M(A) : \tau_i(a^* a) < \infty, i = 1, 2, \ldots, k + 1\} \). Define

\[
h_n(\tau_i) = \min\{2^{-n-1}, \theta([f_n])(\tau_i)\}, \quad i = 1, 2, \ldots, k,
\]

and

\[
h_n(\tau_{k+1}) = \frac{1}{2} \theta([f_n])(\tau_{k+1}).
\]

Define \( h_n(\tau) = h_n(\tau_i) = \bar{h}_n(\tau) \) for \( \tau \in F_k \), \( h_n(t) = \inf\{h_n(\tau) : \tau \in F_k\} \), \( \bar{h}_n(t) = \sup\{h_n(\tau) : \tau \in F_k\} \), for \( t \in S \setminus F_k \). It is easily verified that \( h_n \) is upper semicontinuous and convex while \( \bar{h}_n \) is lower semicontinuous and concave. By [1, Theorem II.3.10] there exists a real affine continuous function \( g_n^* \) on \( S \) such that
0 < \frac{h_n}{g_n} \leq \frac{\tau}{h_n}.\] Hence \(g'_n|F_k = h_n\). Since \(\theta(G)\) is dense in \(\text{Aff}(S)\), for each \(n\) there is \(g_n \in \theta(G)\) such that
\[
|g_n(t) - \frac{1}{2}g'_n(t)| < \frac{1}{4} \inf\{h_n(\tau) : \tau \in F_k\}
\]
for all \(t \in S\). Consequently, we have projections \(p_n \in A\) such that \(p_n \leq f_n, \tau_i(p_n) \leq 2^{-n}, i = 1, 2, \ldots, k\) and \(\tau_{k+1}(p_n) \geq \frac{1}{8} \tau_{k+1}(f_n)\). Set \(p = \sum_{n=1}^{\infty} p_n\). Then \(p \in M(A)\). It is easily verified that \(p \in J_k\) but \(p \notin J_{k+1}\) (just as \(1 \notin J_\tau\) in Lemma 1). Thus \(J_k \supsetneq J_{k+1}\). This completes the proof.

REMARK. Let \(A\) be a nonunital, nonelementary, separable simple \(AF\) \(C^*\)-algebra without continuous scale, such that \(E(S)\) is infinite. If furthermore, \(E(S)\) is closed or, equivalently, \(S\) is a Bauer simplex, then every real continuous function on \(E(S)\) can be extended to a function in \(\text{Aff}(S)\). Therefore an argument similar to that used in this paper shows that \(M(A)/A\) has infinitely many closed ideals. We believe that \(M(A)/A\) has infinitely many closed ideals even if \(E(S)\) is not closed. However, if \(S\) is a general Choquet simplex, a continuous function on \(E(S)\) may not extend to a continuous affine function on \(S\), and this creates a technical problem. Other methods may be needed.

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