PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 1, September 1988

NORMS OF FREE OPERATORS

MASSIMO A. PICARDELLO AND TADEUSZ PYTLIK

(Communicated by John B. Conway)

ABSTRACT. We give a short and elementary proof of the formula for the norm of a free convolution operator on L^2 of a discrete group. The formula was obtained in 1976 by C. Akemann and Ph. Ostrand, and by several other authors afterwards.

1. Introduction. In [**AO**], C. Akemann and Ph. Ostrand carried out an exact computation of the norm, as a convolution operator on l^2, of a function supported on words of length 1 in a free group F, and derived some interesting consequences. Previous estimates had been obtained in [**Le** and **Bo**]. A shorter proof, based upon probabilistic tools, was recently given in [**Wo**]. In the particular (and easier) case of *radial* convolution operators, this result goes back to [**Ke**], and was afterwards obtained in [**Ca**, **Py1**, **FP**, **Py2**]. The approach of the latter papers makes use of spherical functions and the Poisson kernel on F. The same tools, extended to the context of nonradial nearest neighbour symmetric random walks on F, have been used to give an alternative computation (somewhat related to that of [**Wo**]) of the norm of a *selfadjoint* convolutor supported on words of length 1, and also a description of its spectrum [**A**, **St**, **FS**].

On the other hand, a bound for the norm of a general convolution operator supported on words of length n has been established by U. Haagerup in [**Ha**]. This result, which for $n = 1$ is slightly weaker than in [**AO**], follows by a clever use of the combinatorial properties of words in a free group, which reduces the proof to a repeated application of Schwarz inequality.

In this note, we give a very short and elementary proof of the theorem of Akemann and Ostrand. Although this proof is only slightly shorter than in [**Wo**], it is more natural from the point of view of spectral theory. In particular, the estimate from above makes only use of Schwarz inequality, as in [**Py2**]: the estimate from below, modeled upon [**St** and **FS**], studies the action of the operator onto suitable exponential functions which behave as eigenfunctions of its square out of the identity. Although our approach is inspired by [**St** and **FS**], it also applies to nonselfadjoint convolution operators. In order to make our presentation 'reference free' and to avoid mentioning "words of length 1", in the sequel we shall consider, as in [**AO** and **Wo**], the (essentially equivalent) case of convolution operators in a discrete group whose supports satisfy a suitable freedom condition, that we will shortly denote as "free operators".

Received by the editors September 30, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47A30; Secondary 22D25, 43A15, 20E05.

Key words and phrases. Convolution operator, Leinert property, free group, Schwarz inequality.

In particular, the theorem holds also for the direct product of copies of the two-element group $\mathbf{Z}_2$, considered in [**St** and **FS**]: on the other hand, as the elements of length 1 are involutions, all convolutors supported therein are selfadjoint in this particular case, and the result is simpler.

2. The norm of free operators. A *free convolution operator* on a discrete group G is defined, as in [**AO**], as (left) convolution by a function f whose support satisfies the following freedom condition (called the *Leinert condition* in [**AO**]): for every finite sequence $x_1 \cdots x_{2n} \in \operatorname{supp}(f)$ such that $x_i \neq x_{i+1}$ $\forall i$, $x_1 x_2^{-1} x_3 \cdots x_{2n-1} x_{2n}^{-1} \neq e$.

We want to give a short proof of the following result of Akemann and Ostrand [**AO**, Theorem IV.G]:

THEOREM. *The norm of the free operator $\lambda(f)$ is given by*

$$\|\lambda(f)\| = \min_{s \geq 0} \left(2s + \sum_x \left(\sqrt{s^2 + |f(x)|^2} - s \right) \right).$$

Observe that it is enough to restrict attention to the case of convolution by a function supported on words of length one in a free group. Indeed, it is easy to see that the norm as a convolution operator is translation invariant, and is the same on every subgroup which contains the support. Therefore it is enough to embed G in a larger group, where the support is a translate of a free set. We shall not consider the trivial case of convolutors supported on a singleton, that is, translations.

3. The estimate from above. As observed above, we may assume that f is supported on words of length 1 in F. Moreover, we can also assume $f \geq 0$, because $\|\lambda(f)\| \leq \|\lambda(|f|)\|$. For $x \in \operatorname{supp}(f)$, $y \in F$ and $s \geq 0$, define

$$p(x, y; s) = \frac{\sqrt{s^2 + |f(x)|^2} \pm s}{f(x)},$$

where the sign is understood to be minus if there is no cancellation in the word $x^{-1}y$ (that is, $|x^{-1}y| = |x| + |y|$), and plus if there is a cancellation. (See [**A**, **St**, **FS**] for the link between the function p and the resolvent of $\lambda(f)$.) Actually, the absolute value is not needed here, because $f \geq 0$, but is useful to keep notation consistent with §4. Observe that, for a fixed $y \in G$, the sign in the expression for $p(x, y; s)$ is always minus except once (and always minus if $y = e$). Therefore the sum $\sum_x f(x) p(x, y; s)$ is bounded by

$$c(s) = 2s + \sum_x \left(\sqrt{s^2 + |f(x)|^2} - s \right)$$

(in fact the sum is equal to $c(s)$ unless $y = e$).

We are not ready to estimate the convolution norm of f. Following [**Py2**], we shall apply the Schwarz (or Jensen) inequality $\left(\sum_i \alpha_i \beta_i\right)^2 \leq \left(\sum_i \alpha_i\right) \cdot \left(\sum_j \alpha_j \beta_j^2\right)$, for all α_i, $\beta_i \geq 0$. For every complex function h with finite support, one has

$$\begin{aligned} |f * h(y)|^2 &= \left| \sum_x f(x) p(x, y; s) p^{-1}(x, y; s) h(x^{-1}y) \right|^2 \\ &\leq \left(\sum_x f(x) p(x, y; s) \right) \left(\sum_x f(x) p^{-1}(x, y; s) |h(x^{-1}y)|^2 \right). \end{aligned}$$

Therefore

$$\|f * h\|_2^2 \le c(s) \sum_{y,x} f(x) p^{-1}(x, xy; s) |h(y)|^2.$$

Now

$$p^{-1}(x, xy; s) = \frac{\sqrt{s^2 + |f(x)|^2} \mp s}{f(x)},$$

and, as before, $\sum_x f(x) p^{-1}(x, xy; s) \le c(s)$. Thus $\|f * h\|_2^2 \le c(s)^2 \|h\|_2^2$.

4. The estimate from below. In this section we want to show that $\|\lambda(f)\| \ge \min\{c(s) \colon s \ge 0\}$. For this purpose, we do not want any longer to assume that $f \ge 0$ (although we could, by arguing as in [**AO**, Theorem III.G]). Thus f will be a general complex function supported on a Leinert set in G.

For x in G and $s \ge 0$, define $\varsigma(x, s) = 0$ if $f(x) = 0$ and

$$\varsigma(x, s) = \frac{\sqrt{s^2 + |f(x)|^2} - s}{f(x)}$$

otherwise (with notation as in §3, $\varsigma(x, s) = p(x, e; s)$ if $x \in \operatorname{supp}(f)$). Furthermore, denote by $y_1 \cdots y_n$ the reduced expression of a generic word y of length n in the elements of $\operatorname{supp}(f)$ (as already observed, we may also regard $y_1 \cdots y_n$ as the reduced decomposition of a word y in a free group F). Finally, define two functions ϕ_s, ψ_s on G by the recursive rules (given with respect to the length in the free group generated by the support of f):

$$\begin{aligned} \phi_s(e) &= \psi_s(e) = 1, \\ \phi_s(y) &= \varsigma(y_1^{-1}, s) \psi_s(y_2 \cdots y_n), \\ \psi_s(y) &= \overline{\varsigma(y_1, s)} \phi_s(y_2 \cdots y_n). \end{aligned}$$

Observe that $\phi_s = \psi_s$ if $\lambda(f)$ is selfadjoint, i.e. $f(x^{-1}) = \overline{f(x)}$. The following identity holds: for $y \ne e$, $f * \phi_s(y) = c(s) \psi_s(y)$. Indeed,

$$\begin{aligned} f * \phi_s(y) &= f(y_1) \phi_s(y_2 \cdots y_n) + \sum_{x \ne y_1} f(x) \phi_s(x^{-1} y_1 \cdots y_n) \\ &= \psi_s(y) \left(\frac{f(y_1)}{\overline{\varsigma(y_1, s)}} + \sum_{x \ne y_1} f(x) \varsigma(x, s) \right) = c(s) \psi_s(y). \end{aligned}$$

On the other hand, $f * \phi_s(e) = c(s) - 2s$. It follows that

$$\|f * \phi_s\|_2^2 = |f * \phi_s(e)|^2 + c(s)^2 (\|\psi_s\|_2^2 - \psi_s(e)^2) = c(s)^2 \|\psi_s\|_2^2 + 4s(s - c(s)). \tag{1}$$

Let s_0 be the minimum point of $c(s)$ for $s \ge 0$. Excluding the trivial case of convolution operators whose support is a singleton (i.e., translations), we have that s_0 is the only extremum of c, that is, the only solution of $c'(s) = 0$. For $s > s_0$, we claim that $\phi_s, \psi_s \in l^2(G)$ and $\|\phi_s\|_2^2 = \|\psi_s\|_2^2 = 2/c'(s)$. Then the theorem follows by (1), because

$$\|\lambda(f)\|^2 \ge \lim_{s \downarrow s_0} \frac{\|f * \phi_s\|_2^2}{\|\phi_s\|_2^2} = \lim_{s \downarrow s_0} (c(s)^2 + 2s(s - c(s)) c'(s)) = c(s_0)^2.$$

Thus it remains to prove the claim. It is obvious that $\|\phi_s\|_2 = \|\psi_s\|_2$, because

$$(2)\qquad \sum_{|y|=n} |\phi_s(y)|^2 = \sum_{|y|=n} |\psi_s(y)|^2 = \sum_{y_1} |\varsigma(y_1, s)|^2 \sum_{y_2 \neq y_1} |\varsigma(y_2, s)|^2 \cdots \sum_{y_n \neq y_{n-1}} |\varsigma(y_n, s)|^2.$$

For any x of length 1, we define by recurrence a sequence of positive numbers $\alpha_n(s)$ as follows:

$$\alpha_0(x) = 1, \quad \alpha_{n+1}(x) = \sum_{\substack{|y|=1\\ y \neq x}} |\varsigma(y, s)|^2 \alpha_n(y).$$

If $|x| = 1$, the quantity $\alpha_{n+1}(x) + |\varsigma(x, s)|^2 \alpha_n(x)$ is independent of x, and, by (2), is equal to $\sum_{|y|=n+1} |\phi_s(y)|^2$. Thus

$$(3)\qquad \|\phi_s\|_2^2 = (1 + |\varsigma(x, s)|^2) \sum_{n=0}^{\infty} \alpha_n(x).$$

On the other hand, again by (2), $\sum_{|y|=n+1} |\phi_s(y)|^2 = \sum_x |\varsigma(x, s)|^2 \alpha_n(x)$. Therefore,

$$(4)\qquad \|\phi_s\|_2^2 = 1 + \sum_x |\varsigma(x, s)|^2 \sum_{n=0}^{\infty} \alpha_n(x) = 1 + \sum_x \frac{|\varsigma(x, s)|^2}{1 + |\varsigma(x, s)|^2} \|\phi_s\|_2^2.$$

We now compute the last sum. One has

$$\frac{|\varsigma(x, s)|^2}{1 + |\varsigma(x, s)|^2} = \frac{(\sqrt{s^2 + |f(x)|^2} - s)^2}{|f(x)|^2 + (\sqrt{s^2 + |f(x)|^2} - s)^2} = \frac{\sqrt{s^2 + |f(x)|^2} - s}{(\sqrt{s^2 + |f(x)|^2} + s) + (\sqrt{s^2 + |f(x)|^2} - s)} = \frac{\sqrt{s^2 + |f(x)|^2} - s}{2\sqrt{s^2 + |f(x)|^2}}.$$

Thus

$$(5)\qquad \sum_x \frac{|\varsigma(x, s)|^2}{1 + |\varsigma(x, s)|^2} - 1 - \frac{c'(s)}{2}.$$

If $s > s_0$, $c'(s) > 0$, and the last sum is less than 1. By the same argument which yields (3) and (4), we have:

$$(3')\qquad \sum_{|y| \leq N} |\phi_s(y)|^2 \geq (1 + |\varsigma(x, s)|^2) \sum_{n=0}^{N-1} \alpha_n(x),$$

$$(4')\qquad \sum_{|y| \leq N} |\phi_s(y)|^2 = 1 + \sum_x |\varsigma(x, s)|^2 \sum_{n=0}^{N-1} \alpha_n(x) \leq 1 + \sum_x \frac{|\varsigma(x, s)|^2}{1 + |\varsigma(x, s)|^2} \sum_{|y| \leq N} |\phi_s(y)|^2.$$

The last inequality shows that $\|\phi_s\|_2 < \infty$, so the claim follows from (4) and (5).

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DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DELL'AQUILA, L'AQUILA, ITALY

INSTYTUT MATEMATYCZNY, UNIWERSYTET WROCLAWSKY, WROCLAW, POLAND