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OPTIMAL PARTITIONING OF A MEASURABLE SPACE

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ABSTRACT. An α-optimal partition of a measurable space according to n nonatomic probability measures is defined. A minmax theorem is used to find a method of obtaining the α-optimal partition. An application to a problem of fair division is given.

1. Introduction. Let $(\mathscr{X},\mathscr{B})$ be a measurable space and let $\mu_1, \mu_2, \ldots, \mu_n$ denote nonatomic probability measures defined on the same σ-algebra $\mathscr{B}$. By an ordered partition $P = \{A_i\}_{i=1}^n$ of the measurable space $(\mathscr{X},\mathscr{B})$ is meant a collection of disjoint subsets $A_1, A_2, \ldots, A_n$ of $\mathscr{X}$ satisfying $A_i \in \mathscr{B}$ for all $i \in I := \{1, 2, \ldots, n\}$ and $\bigcup_{i=1}^n A_i = \mathscr{X}$. Let $\mathscr{P}$ denote the set of all measurable partitions $P = \{A_i\}_{i=1}^n$ of $\mathscr{X}$. Suppose that $\alpha \in S := \{s = (s_1, s_2, \ldots, s_n) \in R: s_i > 0$ for all $i \in I$ and $\sum_{i=1}^n s_i = 1\}$.

DEFINITION. A partition $P^\alpha = \{A_i^\alpha\}_{i=1}^n \in \mathscr{P}$ is considered to be an α-optimal if

$$\min_{i\in I}[\alpha_i^{-1}\mu_i(A_i^\alpha)] = \sup\{\min_{i\in I}[\alpha_i^{-1}\mu_i(A_i)]: P = \{A_i\}_{i=1}^n \in \mathscr{P}\},$$

where α_i is the ith coordinate of $\alpha \in S$.

It is easy to see that the α-optimal partition is also Pareto optimal (cf. [**8**]).

The problem of α-optimal partitioning of a measurable space $(\mathscr{X},\mathscr{B})$ can be interpreted as the well-known problem of fair division of an object $\mathscr{X}$ (e.g. a cake) among n participants (cf. [**2, 5**]). Here, each μ_i, $i \in I$, represents the individual evaluation of sets from $\mathscr{B}$. We also assume in this problem that $\mu_1, \mu_2, \ldots, \mu_n$ are nonatomic probability measures. Dividing the object $\mathscr{X}$ fairly we are interested in giving the ith person a set $A_i \subset \mathscr{X}$ such that $\mu_i(A_i) \geq 1/n$ for all $i \in I$.

A simple and well-known method for realizing a fair division (of a cake) for two players is "for one to cut, the other to choose". Steinhaus in 1944 (cf. [**5**]) asked whether the fair procedure could be found for dividing a cake among n participants for $n > 2$. He found a solution for $n = 3$ and Banach and Knaster (cf. [**5**]) showed that the solution for $n = 2$ could be extended to arbitrary n. Moreover, Knaster [**5**] proved that if we assume that $\mu_k \neq \mu_j$ for some $k \neq j$, then there exists a partition $P = \{A_i\}_{i=1}^n$ such that $\mu_i(A_i) > 1/n$ for all $i \in I$. Urbanik [**8**] showed the existence of a partition which maximizes $\min_{i\in I}[\mu_i(E_i)]$ over all partitions $P = \{E_i\}_{i=1}^n \in \mathscr{P}$. It is easy to verify that this partition is the α-optimal if $\alpha_i = 1/n$ for each $i \in I$.

Dubins and Spanier [**2**] obtained for arbitrary α the existence of a partition $P = \{A_i\}_{i=1}^n$ such that $\mu_i(A_i) \geq \alpha_i$ ($\mu_i(A_i) > \alpha_i$ if $\mu_k \neq \mu_j$ for some $k \neq j$) for each $i \in I$. The α-optimal partition P^α defined above not only satisfies the

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condition $\mu_i(A_i) \geq \alpha_i$ for $i \in I$, but also is proportionally optimal according to the shares $\alpha_1, \alpha_2, \ldots, \alpha_n$ (cf. [8]). Dubins and Spanier showed additionally that there exists a partition which maximizes $\sum_{i=1}^n \mu_i(A_i)$ and that there is a partition which is optimal in a lexicographic sense. Elton et al. [4] gave an estimation of the number

$$v^* = \sup\left\{\min_{i\in I}[\mu_i(A_i)]: P = \{A_i\}_{i=1}^n \in \mathscr{P}\right\}$$

by suitable inequalities. Their result was improved and generalized for α-optimal partitions by Legut [6] and Wilczyński [9].

Our main purpose in this paper is to show how the α-optimal partition can be obtained from the theorem of Dvoretzky et al. [3].

2. Main Theorem. Under the assumptions given above, Dvoretzky et al. [3] (cf. [2]) proved the following

THEOREM 1. *Let $\overline{\mu}: \mathscr{P} \to R^n$ denote the division vector valued function defined by*

$$\overline{\mu}(P) = (\mu_1(A_1), \mu_2(A_2), \ldots, \mu_n(A_n)) \in R^n, \qquad P = \{A_i\}_{i=1}^n \in \mathscr{P}.$$

Then the range $\overline{\mu}(\mathscr{P})$ of $\overline{\mu}$ is convex and compact in R^n.

To prove our main theorem we make use of the well-known result of Sion (see [1]) which we mention here in the less general form.

THEOREM 2. *Let A and B be convex compact sets. Assume that $K(a,b)$ is a continuous function on the Cartesian product $A \times B$ concave in a and convex in b. Then there exists a point $(a_0, b_0) \in A \times B$ such that*

$$\begin{aligned}\sup_{a\in A}\inf_{b\in B} K(a,b) &= \inf_{b\in B} K(a_0,b) = K(a_0,b_0)\\ &= \sup_{a\in A} K(a,b_0) = \inf_{b\in B}\sup_{a\in A} K(a,b).\end{aligned}$$

Without loss of generality we may assume throughout that $\mu_1, \mu_2, \ldots, \mu_n$ are absolutely continuous with respect to the same measure v (e.g. $v = \sum_{i=1}^n \mu_i$). For all $i \in I$, let $f_i = d\mu_i/dv$. With every $p \in \overline{S}$, where $\overline{S}$ stands for the closure of S in R^n, we will associate $\mathscr{B}$-measurable subsets $B_1(p), B_2(p), \ldots, B_n(p)$ and $C_1(p)$, $C_2(p), \ldots, C_n(p)$ of $\mathscr{X}$ defined by

$$B_i(p) = \bigcap_{\substack{j=1\\ j\neq i}}^{n} \{x \in \mathscr{X}: p_i\alpha_i^{-1} f_i(x) > p_j\alpha_j^{-1} f_j(x)\},$$

$$C_i(p) = \bigcap_{j=1}^{n} \{x \in \mathscr{X}: p_i\alpha_i^{-1} f_i(x) \geq p_j\alpha_j^{-1} f_j(x)\} \qquad \text{for } i \in I.$$

Now we may state the main result of the paper.

THEOREM 3. *For all $\alpha \in S$ there exist a point $p^\alpha = (p_1^\alpha, \ldots, p_n^\alpha) \in \overline{S}$ and a corresponding partition $P^\alpha = \{A_i^\alpha\}_{i=1}^n \in \mathscr{P}$ which satisfies*

(i) $B_i(P^\alpha) \subset A_i^\alpha \subset C_i(p^\alpha)$,

(ii) $\mu_1(A_1^\alpha)/\alpha_1 = \mu_2(A_2^\alpha)/\alpha_2 = \cdots = \mu_n(A_n^\alpha)/\alpha_n$ *and is α-optimal. Moreover, any partition satisfying* (i) *and* (ii) *is α-optimal.*

PROOF. It is obvious that

$$\sup_{a\in\overline{\mu}(\mathscr{P})} \min_{i\in I} \alpha_i^{-1} a_i = \sup_{a\in\overline{\mu}(\mathscr{P})} \min_{p\in\overline{S}} \sum_{i=1}^n p_i \alpha_i^{-1} a_i,$$

where a_i is the ith coordinate of $a \in \overline{\mu}(\mathscr{P})$. Moreover, it follows from Theorem 2 that there exists a point $(p^\alpha, a^\alpha) \in \overline{S} \times \overline{\mu}(\mathscr{P})$ for which

$$\begin{aligned}\sup_{a\in\overline{\mu}(\mathscr{P})} \min_{p\in\overline{S}} \sum_{i=1}^n p_i\alpha_i^{-1}a_i &= \min_{p\in\overline{S}} \sum_{i=1}^n p_i\alpha_i^{-1}a_i^\alpha = \sum_{i=1}^n p_i^\alpha\alpha_i^{-1}a_i^\alpha \\ &= \sup_{a\in\overline{\mu}(\mathscr{P})} \sum_{i=1}^n p_i^\alpha \alpha_i^{-1} a_i = \min_{p\in\overline{S}} \sup_{a\in\overline{\mu}(\mathscr{P})} \sum_{i=1}^n p_i\alpha_i^{-1}a_i.\end{aligned} \tag{1}$$

It is clear that any partition $P^\alpha = \{A_i^\alpha\}_{i=1}^n \in \mathscr{P}$ with $\mu_i(A_i^\alpha) = a_i$ for $i \in I$ is α-optimal. Now, since by (1)

$$\sum_{i=1}^n p_i^\alpha \alpha_i^{-1} \mu_i(A_i^\alpha) = \sup_{P\in\mathscr{P}} \sum_{i=1}^n p_i^\alpha\alpha_i^{-1}\mu_i(A_i)$$

and since

$$\min_{p\in\overline{S}} \sup_{P\in\mathscr{P}} \sum_{i=1}^n p_i\alpha_i^{-1}\mu_i(A_i) = \min_{p\in\overline{S}} \sum_{i=1}^n p_i\alpha_i^{-1}\mu_i(A_i^\alpha)$$

it follows that (i) and (ii) must be fulfilled, respectively. This completes the proof of the first part of the Theorem. The proof of the second part is straightforward.

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