ZONOIDS WITH MINIMAL VOLUME-PRODUCT—
A NEW PROOF

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ABSTRACT. A new and simple proof of the following result is given: The product of the volumes of a symmetric zonoid $A$ in $\mathbb{R}^n$ and of its polar body is minimal if and only if $A$ is the Minkowski sum of $n$ segments.

Introduction. Let $A$ be a convex symmetric body in $\mathbb{R}^n$ and $A^*$ its polar body, with respect to some scalar product. It is well known that the product $P(A) = |A||A^*|$ of the volumes of $A$ and $A^*$ does not depend on the choice of the particular scalar product. It is called the volume-product of $A$. It was proved by Santaló [9] (cf. also [8] for the equality case) that $P(A) \leq P(B_2^n)$ where $B_2^n$ is the Euclidean ball. Bourgain and Milman [1] proved that there exists a universal constant $c > 0$ such that $P(A) \geq c^n P(B_2^n)$. For some particular classes of convex symmetric bodies in $\mathbb{R}^n$, a sharper estimate for the lower bound of $P(A)$ has been obtained. If $A$ is the unit ball of a normed $n$-dimensional space with a 1-unconditional basis, Saint-Raymond [8] proved that $P(A) \geq 4^n/n!$; the equality case, obtained for $1-\infty$ spaces, is discussed in [3 and 7] (cf. also [4]). When $A$ is a zonoid it was proved by Reisner that the same inequality holds, with equality if and only if $A$ is an $n$-cube (or a parallelogram). The proof, given in [5 and 6] depends on probabilistic arguments involving random hyperplanes and a sharp result on the number of vertices of some polyhedra. The aim of this paper is to give a new, simpler and almost self-contained proof.

Notations and definitions. The Lebesgue measure on $\mathbb{R}^n$ is denoted by $dx$. The unit sphere in $\mathbb{R}^n$ is denoted by $S_{n-1} = \{x = (x_i)_{i=1}^n \in \mathbb{R}^n; \sum_{i=1}^n |x_i|^2 = 1\}$.

If $C$ is a compact subset of a $k$-dimensional subspace $E$ of $\mathbb{R}^n$, $|C|$ is its volume with respect to the Lebesgue measure induced in $E$ by $dx$. For subsets $C, D$ of $\mathbb{R}^n$ and $\lambda \geq 0$, let $\lambda C = \{\lambda x; x \in C\}$ and $C + D = \{x + y; x \in C, y \in D\}$ be the Minkowski product and sum. For $x \in \mathbb{R}^n$ let $[-x, x] = \{\alpha x; -1 \leq \alpha \leq 1\}$. Let $A$ be a symmetric (with respect to the origin) convex body in $\mathbb{R}^n$; we say that $A$ is an $n$-cube if $A = \sum_{i=1}^n [-x_i, x_i]$ for some independent vectors $\{x_i\}_{i=1}^n$ in $\mathbb{R}^n$.

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Let $A^*$ be the polar body of $A$ with respect to the canonical scalar product, denoted by $\langle \cdot, \cdot \rangle$ and let $\| \cdot \|_{A^*}$ be the norm associated with $A^*$, that is

$$A^* = \{ y \in \mathbb{R}^n; |\langle x, y \rangle| \leq 1 \text{ for every } x \in A \},$$

$$\|y\|_{A^*} = \max \{ \langle x, y \rangle; x \in A \}.$$

For $x \in S_{n-1}$ let $H(x)$ be the hyperplane through 0 orthogonal to $x$ and $P_x^\perp$ the orthogonal projection onto $H(x)$. Let $A(x) = A \cap H(x)$. We have $A(x) \subset P_x^\perp A = \{ z \in H(x); z + \lambda x \in A \text{ for some } \lambda \in \mathbb{R} \}$. By the bipolar theorem, it is clear that for any $x \in S_{n-1}$ ($P_x^\perp A)^* = A^*(x)$, taking the polar of $P_x^\perp A$ with respect to $\langle \cdot, \cdot \rangle$ on $H(x)$.

A zonoid in $\mathbb{R}^n$ is a convex body for which $\| \cdot \|_{A^*}$ is given by

$$\|y\|_{A^*} = \frac{1}{2} \int_{S_{n-1}} |\langle x, y \rangle| \, d\mu(x), \quad y \in \mathbb{R}^n;$$

here $\mu$ is some positive, even Borel measure on $S_{n-1}$. The measure $\mu$ which is unique, is called the supporting measure of $A$. We have $A = \frac{1}{2} \int_{S_{n-1}} [-x, x] \, d\mu(x)$, where the integration is done by approximating $\mu$ by discrete measures and taking Minkowski sums of segments in $\mathbb{R}^n$. For more details about zonoids we refer to [10].

In dealing with a zonoid in $\mathbb{R}^n$ we always assume that it has nonempty interior.

We shall prove the following result:

**THEOREM.** Let $A$ be a zonoid in $\mathbb{R}^n$, then $P(A) \geq 4^n/n!$, with equality if and only if $A$ is an $n$-cube.

**LEMMA 1.** Let $A$ be a zonoid in $\mathbb{R}^n$ with supporting measure $\mu$. Then

$$(n+1)|A| \int_{S_{n-1}} \left[ \int_{A^*} |\langle x, y \rangle| \, dy \right] \, d\mu(x) = 2|A^*| \int_{S_{n-1}} |P_x^\perp A| \, d\mu(x).$$

In particular, for some $x_0 \in S_{n-1}$ we have

$$(n+1)|A| \int_{A^*} |\langle x_0, y \rangle| \, dy \geq 2|A^*| |P_{x_0}^\perp A|.$$

**PROOF.** By the Fubini theorem,

$$\int_{S_{n-1}} \left[ \int_{A^*} |\langle x, y \rangle| \, dy \right] \, d\mu(x) = \int_{A^*} \left[ \int_{S_{n-1}} |\langle x, y \rangle| \, d\mu(x) \right] \, dy$$

$$= 2 \int_{A^*} \|y\|_{A^*} \, dy = 2 \int_{A^*} \left[ \int_{0}^{\|y\|_{A^*}} \, dt \right] \, dy$$

$$= 2 \int_{0}^{1} \{ y \in \mathbb{R}^n; t \leq \|y\|_{A^*} \leq 1 \} \, dt = \frac{2n}{n+1} |A^*|.$$

By the volume formula for zonoids (cf. [10]) we have $|A| = n^{-1} \int_{S_{n-1}} |P_x^\perp A| \, d\mu(x)$. The first formula is thus proved. The existence of $x_0$ follows, since $\mu$ is a positive measure.

The following lemma is a particular case of a result of [11, p. 182].
Lemma 2. Let \( f : R^+ \rightarrow R^+ \) satisfy: \( f(0) = 1, \int_0^\infty f(x) \, dx > 0 \) and for some \( p > 0, f^{1/p} \) is concave on \( \{ x; f(x) \neq 0 \} \). Then \( \int_0^\infty tf(t) \, dt \leq \frac{p+1}{p+2} [ \int_0^\infty f(t) \, dt ]^2 \) with equality if and only if, for some \( a > 0, f(t) = (1 - at)^p \).

Proof. Let \( a > 0 \) be such that
\[
\int_0^\infty f(t) \, dt = \int_0^\infty (1 - at)^p \, dt = \frac{1}{a(p+1)}.
\]
Let \( g(x) = f(x) - (1 - ax)^p \). Since \( g(0) = 0, \int_0^\infty g(x) \, dx = 0 \) and \( f^{1/p} \) is concave, there exists \( x_0 \geq 0 \) such that \( g(x) \geq 0 \) for \( 0 \leq x \leq x_0 \), and \( g(x) \leq 0 \) for \( x_0 \leq x \). It follows that \( \int_0^\infty g(t) \, dt \leq 0 \) for all \( x \in R^+ \) and thus that
\[
\int_0^\infty tf(t) \, dt = \int_0^\infty \left[ \int_x^\infty f(t) \, dt \right] \, dx \leq \int_0^\infty \left[ \int_x^\infty (1 - at)^p \, dt \right] \, dx = \frac{1}{(p+1)(p+2)a^2} = \frac{p+1}{p+2} \left[ \int_0^\infty f(t) \, dt \right]^2.
\]
There is equality if and only if \( \int_0^\infty f(t) \, dt = \int_0^\infty (1 - at)^p \, dt \) for all \( x \in R^+ \), that is, if and only if \( f(t) = (1 - at)^p \) for all \( t \geq 0 \). □

Lemma 3. Let \( B \) be a symmetric convex body in \( R^n \), \( x \in S_{n-1} \) and \( B(x) = \{ y \in B; (x,y) = 0 \} \). Then
\[
\int_B |(x,y)| \, dy \leq \frac{n}{2(n+1)} \frac{|B|^2}{|B(x)|},
\]
with equality if and only if for some \( y \in R^n \), \( B = \text{conv}(y, -y, B(x)) \).

Proof. Let \( g(t) = |\{ y \in B; (x,y) = t \}|, t \in R \). Then \( g(0) = |B(x)|, g \) is even, \( g(t) = 0 \) for \( |t| > \|x\|_B \), and by the Brunn-Minkowski theorem (cf. [2]), \( g^{1/(n-1)} \) is concave in \( (-\|x\|_B, \|x\|_B) \). By the Fubini theorem, \( |B| = 2 \int_0^\infty g(t) \, dt \) and \( \int_B |(x,y)| \, dy = 2 \int_0^\infty t g(t) \, dt \).

Thus by Lemma 2 applied to \( f = g/g(0) \) and \( p = n - 1 \), we get the required inequality. There is an equality if and only if \( g(t)/g(0) = [1 - (t/\|x\|_B)]^{n-1} \) for \( t \geq 0 \). By the Hahn-Banach theorem, there exists \( y \in B \) such that \( |(y,x)| = \|x\|_B \). By convexity and symmetry we get \( B \supseteq B_1 = \text{conv}(y, -y, B(x)) \). In the case of equality, we have
\[
|B| \geq |B_1| = 2/n \|x\|_B |B(x)| = 2 \int_0^\infty g(t) \, dt = |B|,
\]
and thus \( B = B_1 \). The converse is obtained by calculation. □

Proof of the Theorem. We proceed by induction on \( n \). The result is clear for \( n = 1 \). Let \( A \) be a zonoid in \( R^n \), by Lemma 1 and Lemma 3, for some \( x_0 \) in \( S_{n-1} \) we have
\[
2|P_{x_0} A| |A^*| \leq (n + 1) |A| \int_{A^*} |(x_0, y)| \, dy \leq \frac{n |A| |A^*|}{2 |A^*(x_0)|}.
\]
Since \( (A^*(x_0))^* \) may be identified with \( P_{x_0} A \), we get
\[
P(A) \geq (4/n) P(P_{x_0} A).
\]
Now, $P_{x_0}^\perp A$ is a zonoid in $R^{n-1}$, hence by induction $P(P_{x_0}^\perp A) \geq 4^{n-1}/(n-1)!$ which yields $P(A) \geq 4^n/n!$. Now, if $P(A) = 4^n/n!$ then for this $x_0 \in S_{n-1}$, we have $P(P_{x_0}^\perp A) = 4^{n-1}/(n-1)!$. Thus by induction for the equality case, $P_{x_0}^\perp A$ is an $(n-1)$-cube. By the equality case in Lemma 3, we have also $A^* = \text{conv}(y_0, -y_0, A^*(x_0))$ for some $y_0 \in R^n$. Hence $A$ is an $n$-cube. □

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