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ZONOIDS WITH MINIMAL VOLUME-PRODUCT— A NEW PROOF

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ABSTRACT. A new and simple proof of the following result is given: The product of the volumes of a symmetric zonoid A in $\mathbf{R}^n$ and of its polar body is minimal if and only if A is the Minkowski sum of n segments.

Introduction. Let A be a convex symmetric body in R^n and A^* its polar body, with respect to some scalar product. It is well known that the product $P(A) = |A|\,|A^*|$ of the volumes of A and A^* does not depend on the choice of the particular scalar product. It is called *the volume-product of* A. It was proved by Santaló [**9**] (cf. also [**8**] for the equality case) that $P(A) \leq P(B_2^n)$ where B_2^n is the Euclidean ball. Bourgain and Milman [**1**] proved that there exists a universal constant $c > 0$ such that $P(A) \geq c^n P(B_2^n)$. For some particular classes of convex symmetric bodies in R^n, a sharper estimate for the lower bound of $P(A)$ has been obtained. If A is the unit ball of a normed n-dimensional space with a 1-unconditional basis, Saint-Raymond [**8**] proved that $P(A) \geq 4^n/n!$; the equality case, obtained for $1-\infty$ spaces, is discussed in [**3** and **7**] (cf. also [**4**]). When A is a zonoid it was proved by Reisner that the same inequality holds, with equality if and only if A is an n-cube (or a parallelotope). The proof, given in [**5** and **6**] depends on probabilistic arguments involving random hyperplanes and a sharp result on the number of vertices of some polyhedra. The aim of this paper is to give a new, simpler and almost self-contained proof.

Notations and definitions. The Lebesgue measure on R^n is denoted by dx. The unit sphere in R^n is denoted by $S_{n-1} = \{x = (x_i)_{i=1}^n \in R^n; \sum_{i=1}^n |x_i|^2 = 1\}$.

If C is a compact subset of a k-dimensional subspace E of R^n, $|C|$ is its volume with respect to the Lebesgue measure induced in E by dx. For subsets C, D of R^n and $\lambda \geq 0$, let $\lambda C = \{\lambda x; x \in C\}$ and $C + D = \{x + y; x \in C, y \in D\}$ be the Minkowski product and sum. For $x \in R^n$ let $[-x, x] = \{\alpha x; -1 \leq \alpha \leq 1\}$. Let A be a symmetric (with respect to the origin) convex body in R^n; we say that A is an *n-cube* if $A = \sum_{i=1}^n [-x_i, x_i]$ for some independent vectors $\{x_i\}_{i=1}^n$ in R^n.

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Let A^* be the polar body of A with respect to the canonical scalar product, denoted by $\langle\ ,\ \rangle$ and let $\|\ \|_{A^*}$ be the norm associated with A^*, that is

$$A^* = \{y \in R^n; |\langle x, y\rangle| \le 1 \text{ for every } x \in A\},$$
$$\|y\|_{A^*} = \max\{\langle x, y\rangle; x \in A\}.$$

For $x \in S_{n-1}$ let $H(x)$ be the hyperplane through 0 orthogonal to x and $P_x^\perp$ the orthogonal projection onto $H(x)$. Let $A(x) = A \cap H(x)$. We have $A(x) \subset P_x^\perp A = \{z \in H(x); z + \lambda x \in A \text{ for some } \lambda \in R\}$. By the bipolar theorem, it is clear that for any $x \in S_{n-1}$ $(P_x^\perp A)^* = A^*(x)$, taking the polar of $P_x^\perp A$ with respect to $\langle\ ,\ \rangle$ on $H(x)$.

A zonoid in R^n is a convex body for which $\|\ \|_{A^*}$ is given by

$$\|y\|_{A^*} = \frac{1}{2}\int_{S_{n-1}} |\langle x, y\rangle|\, d\mu(x), \qquad y \in R^n;$$

here μ is some positive, even Borel measure on S_{n-1}. The measure μ which is unique, is called the *supporting measure* of A. We have $A = \frac{1}{2}\int_{S_{n-1}}[-x, x]\, d\mu(x)$, where the integration is done by approximating μ by discrete measures and taking Minkowski sums of segments in R^n. For more details about zonoids we refer to [**10**]. In dealing with a zonoid in R^n we always assume that it has nonempty interior.

We shall prove the following result:

THOEREM. *Let A be a zonoid in R^n, then $P(A) \ge 4^n/n!$, with equality if and only if A is an n-cube.*

LEMMA 1. *Let A be a zonoid in R^n with supporting measure μ. Then*

$$(n+1)|A|\int_{S_{n-1}}\left[\int_{A^*}|\langle x, y\rangle|\, dy\right] d\mu(x) = 2|A^*|\int_{S_{n-1}}|P_x^\perp A|\, d\mu(x).$$

In particular, for some $x_0 \in S_{n-1}$ we have

$$(n+1)|A|\int_{A^*}|\langle x_0, y\rangle|\, dy \ge 2|A^*|\,|P_{x_0}^\perp A|.$$

PROOF. By the Fubini theorem,

$$\begin{aligned}\int_{S_{n-1}}\left[\int_{A^*}|\langle x, y\rangle|\, dy\right] d\mu(x) &= \int_{A^*}\left[\int_{S_{n-1}}|\langle x, y\rangle|\, d\mu(x)\right] dy\\ &= 2\int_{A^*}\|y\|_{A^*}\, dy = 2\int_{A^*}\left[\int_0^{\|y\|_{A^*}} dt\right] dy\\ &= 2\int_0^1 |\{y \in R^n; t \le \|y\|_{A^*} \le 1\}|\, dt = \frac{2n}{n+1}|A^*|.\end{aligned}$$

By the volume formula for zonoids (cf. [**10**]) we have $|A| = n^{-1}\int_{S_{n-1}}|P_x^\perp A|\, d\mu(x)$. The first formula is thus proved. The existence of x_0 follows, since μ is a positive measure. $\square$

The following lemma is a particular case of a result of [**11**, p. 182].

LEMMA 2. *Let* $f\colon R_+ \to R_+$ *satisfy:* $f(0) = 1$, $\int_0^\infty f(x)\,dx > 0$ *and for some* $p > 0$, $f^{1/p}$ *is concave on* $\{x; f(x) \neq 0\}$. *Then* $\int_0^\infty t f(t)\,dt \le \frac{p+1}{p+2}[\int_0^\infty f(t)\,dt]^2$ *with equality if and only if, for some* $a > 0$, $f(t) = (1-at)_+^p$.

PROOF. Let $a > 0$ be such that

$$\int_0^\infty f(t)\,dt = \int_0^\infty (1-at)_+^p\,dt = \frac{1}{a(p+1)}.$$

Let $g(x) = f(x) - (1-ax)_+^p$. Since $g(0) = 0$, $\int_0^\infty g(x)\,dx = 0$ and $f^{1/p}$ is concave, there exists $x_0 \ge 0$ such that $g(x) \ge 0$ for $0 \le x \le x_0$, and $g(x) \le 0$ for $x_0 \le x$. It follows that $\int_x^\infty g(t)\,dt \le 0$ for all $x \in R_+$ and thus that

$$\begin{aligned}\int_0^\infty t f(t)\,dt &= \int_0^\infty \left[\int_x^\infty f(t)\,dt\right] dx \le \int_0^\infty \left[\int_x^\infty (1-at)_+^p\,dt\right] dx \\ &= \frac{1}{(p+1)(p+2)a^2} = \frac{p+1}{p+2}\left[\int_0^\infty f(t)\,dt\right]^2.\end{aligned}$$

There is equality if and only if $\int_x^\infty f(t)\,dt = \int_x^\infty (1-at)_+^p\,dt$ for all $x \in R_+$, that is, if and only if $f(t) = (1-at)_+^p$ for all $t \ge 0$. □

LEMMA 3. *Let* B *be a symmetric convex body in* R^n, $x \in S_{n-1}$ *and* $B(x) = \{y \in B; \langle x, y\rangle = 0\}$. *Then*

$$\int_B |\langle x, y\rangle|\,dy \le \frac{n}{2(n+1)} \frac{|B|^2}{|B(x)|}$$

with equality if and only if for some $y \in R^n$, $B = \operatorname{conv}(y, -y, B(x))$.

PROOF. Let $g(t) = |\{y \in B; \langle x, y\rangle = t\}|$, $t \in R$. Then $g(0) = |B(x)|$, g is even, $g(t) = 0$ for $|t| > \|x\|_{B^*}$, and by the Brunn-Minkowski theorem (cf. [**2**]), $g^{1/(n-1)}$ is concave in $(-\|x\|_{B^*}, \|x\|_{B^*})$. By the Fubini theorem, $|B| = 2\int_0^\infty g(t)\,dt$ and $\int_B |\langle x, y\rangle|\,dy = 2\int_0^\infty t g(t)\,dt$.

Thus by Lemma 2 applied to $f = g/g(0)$ and $p = n-1$, we get the required inequality. There is an equality if and only if $g(t)/g(0) = [1 - (t/\|x\|_{B^*})]_+^{n-1}$ for $t \ge 0$. By the Hahn-Banach theorem, there exists $y \in B$ such that $|\langle y, x\rangle| = \|x\|_{B^*}$. By convexity and symmetry we get $B \supset B_1 = \operatorname{conv}(y, -y, B(x))$. In the case of equality, we have

$$|B| \ge |B_1| = 2/n\|x\|_{B^*}|B(x)| = 2\int_0^\infty g(t)\,dt = |B|,$$

and thus $B = B_1$. The converse is obtained by calculation. □

PROOF OF THE THEOREM. We proceed by induction on n. The result is clear for $n = 1$. Let A be a zonoid in R^n, by Lemma 1 and Lemma 3, for some x_0 in S_{n-1} we have

$$2|P_{x_0}^\perp A|\,|A^*| \le (n+1)|A| \int_{A^*} |\langle x_0, y\rangle|\,dy \le \frac{n|A|\,|A^*|}{2|A^*(x_0)|}.$$

Since $(A^*(x_0))^*$ may be identified with $P_{x_0}^\perp A$, we get

$$P(A) \ge (4/n) P(P_{x_0}^\perp A).$$

Now, $P_{x_0}^{\perp}A$ is a zonoid in R^{n-1}, hence by induction $P(P_{x_0}^{\perp}A) \geq 4^{n-1}/(n-1)!$ which yields $P(A) \geq 4^n/n!$. Now, if $P(A) = 4^n/n!$ then for this $x_0 \in S_{n-1}$, we have $P(P_{x_0}^{\perp}A) = 4^{n-1}/(n-1)!$. Thus by induction for the equality case, $P_{x_0}^{\perp}A$ is an $(n-1)$-cube. By the equality case in Lemma 3, we have also $A^* = \operatorname{conv}(y_0, -y_0, A^*(x_0))$ for some $y_0 \in R^n$. Hence A is an n-cube. $\square$

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