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DEFINING JUMP CLASSES IN THE DEGREES BELOW $\mathbf{0}'$[1]

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ABSTRACT. We prove that, for each degree $\mathbf{c}$ r.e. in and above $\mathbf{0}^{(3)}$, the class of degrees $\mathbf{x} \leq \mathbf{0}'$ with $\mathbf{x}^{(3)} = \mathbf{c}$ is definable without parameters in $\mathscr{D}(\leq \mathbf{0}')$, the degrees below $\mathbf{0}'$. Indeed the same definitions work below any r.e. degree $\mathbf{r}$ in place of $\mathbf{0}'$. Thus for each r.e. degree $\mathbf{r}, \mathrm{Th}(\mathscr{D}(\leq \mathbf{r}))$ uniquely determines $\mathbf{r}^{(3)}$.

Introduction. Definability results, along with characterizing the complexity of theories and restricting possible automorphisms, have been among the major goals and successes of degree theory over the past ten years. Although there are a number of important older results (such as Enderton and Putnam [**1970**] and Sacks [**1971**] defining the ω-jump from the Turing jump), the first major breakthrough came with Jockusch and Simpson [**1976**]. They proved the definability of many natural classes of degrees in $\mathscr{D}'$, the Turing degrees with Turing reducibility, $\leq$, and the jump operators, $'$. Simpson [**1977**] next proved that the theory of $\mathscr{D}'$ as well as that of $\mathscr{D}$, the Turing degrees with just $\leq$, is equivalent to $\mathrm{Th}^2(\mathbb{N})$, the theory of true second order arithmetic. The coding methods introduced for this result also gave a new class of definability results for $\mathscr{D}'$ (though not for $\mathscr{D}$): Every relation on degrees above $\mathbf{0}^{(\omega)}$ is definable in $\mathscr{D}'$ iff it is definable in second order arithmetic.

Nerode and Shore [**1979**] and [**1981**] introduced another approach to analyzing the theory of $\mathscr{D}$ which led to results on definability in, and automorphisms of, $\mathscr{D}$ (see also Shore [**1981**]). As coding methods alone can only give definability results from parameters, at this stage, the definability results for $\mathscr{D}$ were only in terms of parameters or predicates for jump ideals. To get outright definability results one must exploit some special property of particular degrees. The key properties here turned out to be iterated relative recursive enumerability (the n-rea degrees for $n \in \omega$) and being a minimal cover. Exploiting the relation between these properties was the key to defining the class of degrees of arithmetic sets in Jockusch and Shore [**1984**]. Combining this result with the coding methods of Nerode and Shore eliminated the need for the jump in Simpson's results on definability. Thus, for example, the ω-jump is definable in $\mathscr{D}$ as are any relations $R(x_1, \ldots, x_n)$ on the degrees that are definable in second order arithmetic which depend only on the $x_i^{(\omega)}$ or more precisely are invariant under joining with arithmetic sets.

Slaman and Woodin [**1986**] gives yet another even simpler approach, using coding by finite extension methods, to proving that the theory of $\mathscr{D}$ is $\mathrm{Th}^2(\mathbb{N})$. Their

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methods also gave new results on definability from parameters: Every countable relation on the degrees definable in second order arithmetic is definable in $\mathscr{D}$ from finitely many parameters.

Other than $\mathscr{D}$, the degree structures of most interest have been $\mathscr{D}(\leq \mathbf{0}')$, the degrees below $\mathbf{0}'$ and $\mathscr{R}$, the recursively enumerable degrees. In the first of these structures the plan of attack on the problems of characterizing the complexity of theories, definability and automorphism have followed the same general line of approach as for the degrees as a whole. The proofs however have, in general, been much more difficult as working below $\mathbf{0}'$ requires significantly more complicated construction techniques. The first major results are in Shore [**1981**] where it is shown that the theory of $\mathscr{D}(\leq \mathbf{0}')$ is equivalent to Th($\mathbb{N}$), the theory of true first order arithmetic. As far as definability results are concerned, that paper contains only a proof that the high degrees, $\mathbf{H}_1$, can be definably separated from $\bar{\mathbf{H}}_3$, the degrees which are not high$_3$. (A degree $\mathbf{x}$ is said to be high$_n$ if $\mathbf{x}^{(n)} = \mathbf{0}^{(n+1)}$. The set of such degrees is denoted by $\mathbf{H}_n$. Similarly we use low$_n$ and $\mathbf{L}_n$ for the degrees $\mathbf{x}$ such that $\mathbf{x}^{(n)} = \mathbf{0}^{(n)}$. We use $\mathbf{H}_n[\mathbf{k}]$ and $\mathbf{L}_n[\mathbf{k}]$ to denote the corresponding classes relativized to $\mathbf{k}$.) Slaman and Woodin [**1986**] also adapted their methods to the degrees below $\mathbf{0}'$ to prove, for example, that the r.e. degrees are definable in $\mathscr{D}(\leq \mathbf{0}')$ but, for course, only from parameters.

For $\mathscr{R}$ the methods of attack are quite different from the other degree structures and results have been even slower to appear. Harrington and Slaman [**1989**] have shown that the theory of $\mathscr{R}$ is equivalent to Th($\mathbb{N}$) but no real results on automorphisms or definability have yet been derived from their coding methods. Slaman and Shore [**1988**] and [**1989**] have, however, definably separated $\mathbf{L}_2$ from $\mathbf{H}_1$ in $\mathscr{R}$ by a direct analysis of particular order theoretic properties of the r.e. degrees.

In this paper we will show that all the jump classes from $\mathbf{L}_3$ to $\mathbf{H}_3$ are definable in $\mathscr{D}(\leq \mathbf{0}')$. In fact, for any $\mathbf{c}$ recursively enumerable in and above (*rea in*) $\mathbf{0}^{(3)}$, the class of degrees $\mathbf{x}$ below $\mathbf{0}'$ such that $\mathbf{x}^{(3)} = \mathbf{c}$ is definable (without parameters) in $\mathscr{D}(\leq \mathbf{0}')$. As with the degrees as a whole we must exploit, in addition to the coding methods of Shore [**1981**], some special properties of certain classes of degrees to get outright definability results. Relative recursive enumerability again plays a role as does the notion of 1-genericity which implies being r.e. in some smaller set (Jockusch [**1980**]). The connection with the jump operator comes through construction techniques available for non-low$_2$ sets (Jockusch and Posner [**1987**]). On the other hand the only property of $\mathbf{0}'$ needed is recursive enumerability. Thus for each r.e. $\mathbf{r}$ and each $\mathbf{c}$ rea in $\mathbf{0}^{(3)}$ the class of degrees $\mathbf{x}$ below $\mathbf{r}$ with $\mathbf{x}^{(3)} = \mathbf{c}$ is definable in $\mathscr{D}(\leq \mathbf{r})$. Thus for r.e. degrees $\mathbf{r}$, the theory of $\mathscr{D}(\leq \mathbf{r})$ uniquely determines $\mathbf{r}^{(3)}$. The best previous result along these lines had been the separation of the high degrees from the non-high$_3$ ones given by Shore [**1981**, Theorem 4.5]. In addition, there have more recently been a number of papers establishing specific examples of order theoretic properties true below some r.e. degrees but not all as in Cooper and Epstein [**1987**], Cooper [**1988**], Slaman and Steel [**1988**] and Slaman [**1989**].

We should also remark that the gap here, and in some of the results on the degrees as a whole, between our results on the level of the triple jump and ones at the level of the Turing jump itself seems unbridgeable by current methods. The crucial barrier arises from the fact that the relation "Turing reducible to A" is

itself a three quantifier (in A) property. Thus if one wants to define the class of $\mathbf{x} \leq \mathbf{0}'$ such that $\mathbf{x}' = \mathbf{c}$ for $\mathbf{c}$ rea in $\mathbf{0}'$ or to tackle the first and possibly most basic definability question of $\mathscr{D}$, the definability of the jump operator (raised originally in Kleene and Post [**1954**]), one must look for a new approach.

1. Preliminaries. Our proof relies heavily on results (usually relativized) of Shore [**1981**] and Lerman [**1983**]. We also use some simpler theorems of Jockusch and Posner [**1978**] and Jockusch [**1980**]. In this section we will collect and describe the results needed (relativizing them if necessary without explicit comment). The proof of our definability results from these facts will be given in the next section. We begin with the coding schemes from Shore [**1981**] and [**1982**].

THEOREM 1.1 (SHORE [**1982**] AND [**1981**, §1]). *There is a definable relation on pairs of degrees* $\mathbf{b} <_T \mathbf{e}$, $[\mathbf{b}, \mathbf{e}]$ *effectively codes a model of arithmetic which provides a translation of arithmetic into the theory of* $\mathscr{D}[\mathbf{b}, \mathbf{e}]$, *the degrees between* $\mathbf{b}$ *and* $\mathbf{e}$, *such that for* $E \in \mathbf{e}$

(a) *The set* S *of indices* i *such that* $\{\{i\}^E : i \in S\} = \{U : U \in \mathbf{h}$ *and* $\mathbf{h}$ *is in the ideal* I *generated by the degrees* $\mathbf{d}_n$ *which represent the natural numbers in the model effectively coded by* $[\mathbf{b}, \mathbf{d}]\}$ *is* Σ_3^E.

(b) *There is a function* $h \leq_T E^{(3)}$ *such that, for each* $n \in \omega$, $\deg(\{h(n)\}^E) = \mathbf{d}_n$ *represents the natural number* n *in the model coded by* $[\mathbf{b}, \mathbf{e}]$.

As, by Shore [**1982**], there are recursively presented lattices with 0 and 1 which satisfy the definition of effectively coding a model of arithmetic (and indeed ones for which the model coded is the standard model), we need the following result to get the existence of enough degree intervals which effectively code a model of arithmetic.

THEOREM 1.2 (LERMAN [**1983**, THEOREM XII.5.12]). *If* $\mathbf{f}$ *is rea but not recursive in* $\mathbf{b}$ *and* $\mathscr{L}$ *is a recursively presented lattice, then there exists an* $\mathbf{e}$ *strictly between* $\mathbf{b}$ *and* $\mathbf{f}$ *such that* $[\mathbf{b}, \mathbf{e}]$ *is isomorphic to* $\mathscr{L}$. *We can also choose* $E \in \mathbf{e}$ *so that there is a recursive function* $k : \mathscr{L} \to \mathbb{N}$ *such that* $x \to \deg(\{k(x)\}^E)$ *gives the required isomorphism.*

Thus in particular, with $\mathbf{b}$ and $\mathbf{f}$ as in the theorem there is an $\mathbf{e}$ and an $E \in \mathbf{e}$ such that $[\mathbf{b}, \mathbf{e}]$ effectively codes the standard model of arithmetic and a recursive function h such that $\deg(\{h(n)\}^E = \mathbf{d}_n$ is the degree representing the natural number n in this model.

Now to guarantee that a model of arithmetic is standard, it suffices to say of the model that each initial segment in a class which includes the set of standard numbers has a last element or is all of the model. The basic lemma needed to definably pick out the intervals that code the standard model of arithmetic is then the following:

LEMMA 1.3 (SHORE [**1981**, LEMMA 4.2]). *Suppose we are given a set* $E \in \mathbf{e}$ *and one* $W \in \Sigma_3^E$ *such that* $I_W = \{\deg(\{i\}^E) : i \in W\}$ *is an ideal in* $[\mathbf{b}, \mathbf{e}]$. *If* $F \in \mathbf{f}$ *is r.e. in and strictly above* $\mathbf{e}$ *then there are* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *in* $[\mathbf{b}, \mathbf{f}]$ *which form an exact pair for* I_W *in* $[\mathbf{b}, \mathbf{f}]$, *i.e.* $I_W = \{\mathbf{x} : \mathbf{b} \leq \mathbf{x} \leq \mathbf{a}_0, \mathbf{a}_1\}$.

Thus if $\mathbf{f}$ is r.e. in and strictly above $\mathbf{b}$, we can definably in $\mathscr{D}[\mathbf{b}, \mathbf{f}]$ pick out the intervals $[\mathbf{b}, \mathbf{e}]$ with $\mathbf{e} \leq \mathbf{f}$ such that $[\mathbf{b}, \mathbf{e}]$ effectively codes a standard model of arithmetic.

We must next discuss the coding of sets in such models of arithmetic.

DEFINITION 1.4. Suppose $[\mathbf{b}, \mathbf{e}]$ effectively codes the standard model of arithmetic. We say that *the degrees x and y code the set of natural numbers $S(x, y)$ in the model coded by $[b,e]$* when $n \in S(\mathbf{x}, \mathbf{y})$ iff $\mathbf{d}_n$, the degree representing n in this model, is below both $\mathbf{x}$ and $\mathbf{y}$.

LEMMA 1.5. *If $[\mathbf{b}, \mathbf{e}]$ effectively codes the standard model of arithmetic, $F \in \mathbf{f}$ and e, $\mathbf{x}, \mathbf{y} < \mathbf{f}$, then $S(\mathbf{x}, \mathbf{y})$ is recursive in $F^{(3)}$.*

PROOF. By Lemma 1.1(b) there is a function $h \leq_T F^{(3)}$ such that $\mathbf{d}_n = \deg(\{h(n)^F\})$. As the relation of i and j given by $\{i\}^F \leq_T \{j\}^F$ is also recursive in $F^{(3)}$ by a simple quantifier counting argument, the lemma follows from the definition of $S(\mathbf{x}, \mathbf{y})$.

We close this section with the results which establish the special properties of non-low$_2$ sets which give us a handle on outright definability results in $\mathscr{D}(\leq \mathbf{0}')$.

THEOREM 1.6 (JOCKUSCH AND POSNER [**1978**, p. 716]). (i) *If $\mathbf{k} < \mathbf{a} \notin \mathbf{L}_2[\mathbf{k}]$, there is a $\mathbf{b} \in [\mathbf{k}, \mathbf{a}]$ which is 1-generic over $\mathbf{k}$. (The only fact we will need about 1-generics is given in Theorem 1.7.)*

(ii) *If $\mathbf{k} < \mathbf{a} \notin \mathbf{L}_2[\mathbf{k}]$, $\mathbf{c} \geq \mathbf{a} \vee \mathbf{k}'$ and $\mathbf{c}$ is r.e. in $\mathbf{a}$, then there is a $\mathbf{b} \in [\mathbf{k}, \mathbf{a}]$ such that $\mathbf{b}' = \mathbf{c}$.*

THEOREM 1.7 (JOCKUSCH [**1980**, THEOREM 5.1]). *If $\mathbf{a} > \mathbf{k}$ is 1-generic in $\mathbf{k}$, there is a $\mathbf{b} \in [\mathbf{k}, \mathbf{a}]$ such that $\mathbf{a}$ is r.e. in and strictly above $\mathbf{b}$.*

2. Defining the jump classes.

THEOREM 2.1. *For each r.e. degree $\mathbf{r}$ and each $\mathbf{c}$ rea in $\mathbf{0}^{(3)}$ the class of degrees $\mathbf{x} \leq \mathbf{r}$ such that $\mathbf{x}^{(3)} = \mathbf{c}$ is definable in $\mathscr{D}(\leq \mathbf{r})$.*

PROOF. As $\mathbf{c}$ is r.e. in $\mathbf{0}^{(3)}$, there is a formula of arithmetic which defines a set $C \in \mathbf{c}$. Thus given an interval $[\mathbf{b}, \mathbf{e}]$ which effectively codes the standard model of arithmetic, we can definably say of a set $S(\mathbf{x}, \mathbf{y})$ coded by the pair $\mathbf{x}, \mathbf{y} < \mathbf{f}$ that it is of degree $\mathbf{c}$. (Remember, we have all the formulas of arithmetic for the model coded by $[\mathbf{b}, \mathbf{e}]$ available to us definably in the degrees below $\mathbf{f}$ in terms of $\leq, \mathbf{b}$ and $\mathbf{e}$.) We can now give the desired definition.

DEFINITION 2.2. $P(\mathbf{a}, \mathbf{c})$ holds iff $\mathbf{c}$ is the maximum of $\mathbf{0}^{(3)}$ and all the degrees coded by any pair $\mathbf{x}, \mathbf{y}$ below $\mathbf{a}$ in the model corresponding to any $[\mathbf{b}, \mathbf{e}]$ with $\mathbf{e} < \mathbf{a}$ which effectively codes the standard model of arithmetic.

The key points in the verification that, for a fixed $\mathbf{c}$, $P(\mathbf{a}, \mathbf{c})$ is a definable property of degrees in $\mathscr{D}(\leq \mathbf{r})$ are the following:

(1) By Theorem 1.1 "$[\mathbf{b}, \mathbf{e}]$ effectively codes a model of arithmetic" is definable in terms of $\leq$ even within the interval $[\mathbf{b}, \mathbf{e}]$.

(2) If $[\mathbf{b}, \mathbf{e}]$ effectively codes a model of arithmetic and $\mathbf{e} < \mathbf{a}$, then $\mathbf{r}$ is clearly r.e. in and strictly above $\mathbf{e}$. Thus by Lemma 1.3 we can definably say in $\mathscr{D}(\leq \mathbf{r})$ that the model coded by $[\mathbf{b}, \mathbf{e}]$ is standard.

(3) If $[\mathbf{b}, \mathbf{e}]$ effectively codes a model of arithmetic we can define in $\mathscr{D}(\leq \mathbf{r})$ any arithmetic properties of any sets coded in this model by pairs $\mathbf{x}, \mathbf{y}$ below $\mathbf{r}$. This claim follows from the fact (Theorem 1.1) that we have a translation of arithmetic into the theory of $[\mathbf{b}, \mathbf{e}]$ and the definition of coding sets in this model (Definition

1.5) which shows that $n \in S(\mathbf{x},\mathbf{y})$ is definable in terms of $\leq$ from $\mathbf{x},\mathbf{y}$ and the arithmetic structure of the model.

Keeping these points in mind, one can straightforwardly define $P(\mathbf{a},\mathbf{c})$ in $\mathscr{D}(\leq \mathbf{r})$ for each $\mathbf{c}$. We must now show that for $\mathbf{a} \leq \mathbf{r}$ and $\mathbf{c}$ rea in $\mathbf{0}^{(3)}$, $P(\mathbf{a},\mathbf{c}) \Leftrightarrow \mathbf{a}^{(3)} = \mathbf{c}$.

By Lemma 1.5 any set coded by a pair below $\mathbf{a}$ in a model coded by an interval below $\mathbf{a}$ is recursive in $\mathbf{a}^{(3)}$. thus $P(\mathbf{a},\mathbf{c})$ implies that $\mathbf{c} \leq \mathbf{a}^{(3)}$. It is then clear from the definition of $P(\mathbf{a},\mathbf{c})$ that if $\mathbf{a} \in \mathbf{L}_2$, $P(\mathbf{a},\mathbf{c})$ holds iff $\mathbf{c} = \mathbf{0}^{(3)}$ as required. For $\mathbf{a} \notin \mathbf{L}_2$ it now suffices to show that the appropriate maximum is obtained, i.e. there are $\mathbf{b},\mathbf{e},\mathbf{x}$ and $\mathbf{y}$ below $\mathbf{a}$ such that $\mathbf{x}$ and $\mathbf{y}$ code a set of degree $\mathbf{a}^{(3)}$ in the standard model coded by $[\mathbf{b},\mathbf{e}]$.

As $\mathbf{a} \notin \mathbf{L}_2$, i.e. $\mathbf{0}^{(2)} < \mathbf{a}^{(2)}$, we can apply the Robinson jump interpolation theorem (Robinson [**1971**]) in the usual way to choose a $\mathbf{d} \geq \mathbf{a} \vee \mathbf{0}'$ r.e. in $\mathbf{a}$ such that $\mathbf{d}' < \mathbf{a}^{(2)}$ but $\mathbf{d}^{(2)} = \mathbf{a}^{(3)}$. By Lemma 1.6(i) we may then choose a $\mathbf{k} < \mathbf{a}$ such that $\mathbf{k}' = \mathbf{d}$ and so $a \in \mathbf{L}_3[\mathbf{k}] - \mathbf{L}_2[\mathbf{k}]$, i.e. $\mathbf{k}^{(3)} = \mathbf{a}^{(3)}$ but $\mathbf{k}^{(2)} < \mathbf{a}^{(2)}$. By Lemma 1.6(ii) and Lemma 1.7 we may also choose $\mathbf{g}$ and $\mathbf{b}$ such that $\mathbf{g} \in [\mathbf{k},\mathbf{a}]$ is 1-generic over $\mathbf{k}$ and r.e. in $\mathbf{b} \in [\mathbf{k},\mathbf{g}]$.

Now by Theorem 1.2 we may choose an $\mathbf{e} \in [\mathbf{b},\mathbf{g}]$, an $E \in \mathbf{e}$ and a recursive function h such that $[\mathbf{b},\mathbf{e}]$ effectively codes the standard model of arithmetic and $\mathbf{d}_n = \deg(\{h(n)\}^E)$ is the code for n in this model. Note next that the ideal generated in $[\mathbf{b},\mathbf{e}]$ by those $\mathbf{d}_n$ such that $n \in E^{(3)}$ is Σ_3^E in the sense of Lemma 1.3: One can generate every i such that $\deg(\{i\}^E)$ is in this ideal by starting with the Σ_3^E set $\{h(n) \colon n \in E\}$ and closing downward with respect to $\leq_T$ and under join. As join is a recursive operation on indices and we can enumerate the indices i and j such that $\{j\}^E \leq_T \{i\}^E$ recursively in $E^{(2)}$, the set of indices i such that $\deg(\{i\}^E)$ is in this ideal is Σ_3^E as required. Lemma 1.3 now tells us that there are $\mathbf{x}$ and $\mathbf{y}$ below $\mathbf{g}$ that code the set $E^{(3)}$ in the model coded by $[\mathbf{b},\mathbf{e}]$. As $\mathbf{k} < \mathbf{e} < \mathbf{a}$ and $\mathbf{k}^{(3)} = \mathbf{a}^{(3)}$, $E^{(3)} \in \mathbf{a}^{(3)}$. Thus for any $\mathbf{a} \notin \mathbf{L}_2$ we have coded a set, $E^{(3)}$, of degree $\mathbf{a}^{(3)}$ in a standard model coded by an interval $[\mathbf{b},\mathbf{e}]$ below $\mathbf{a}$ by degrees $\mathbf{x}$ and $\mathbf{y}$ which are also below $\mathbf{a}$ as required. □

COROLLARY 2.3. *For each r.e. degree* $\mathbf{r}$, $\mathrm{Th}(\mathscr{D}(\leq \mathbf{r}))$ *uniquely determines* $\mathbf{r}^{(3)}$.

PROOF. $\exists \mathbf{x}(P(\mathbf{x},\mathbf{r}^{(3)}) \& \forall \mathbf{y}(\mathbf{y} \leq \mathbf{x}))$ holds in $\mathscr{D}(\leq \mathbf{s})$ for $\mathbf{s}$ r.e. iff $\mathbf{s}^{(3)} = \mathbf{r}^{(3)}$. □

COROLLARY 2.4. *For each* $n \geq 3$, *the classes* $\mathbf{L}_n$ *and* $\mathbf{H}_n$ *are definable in* $\mathscr{D}(\leq \mathbf{r})$ *for any r.e. degree* $\mathbf{r}$.

PROOF. The proof of the theorem shows that a degree $\mathbf{a} < \mathbf{r}$ is $\mathbf{L}_n(\mathbf{H}_n)$ iff the maximum of the $(n-3)$ jumps of the degrees coded by any pair $\mathbf{x},\mathbf{y}$ below $\mathbf{a}$ in any model, corresponding to $[\mathbf{b},\mathbf{e}]$ with $\mathbf{e} < \mathbf{a}$ which effectively codes a model of arithmetic is $\mathbf{0}^{(n)}$ $(\mathbf{0}^{(n+1)})$. It also shows that these properties are definable in the degrees below $\mathbf{r}$. □

We believe that the definability of the jump classes can be used as a stepping stone to more general results on definability below $\mathbf{0}'$ as was that of the arithmetic degrees in $\mathscr{D}$. In particular we believe that every relation $R(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ on degrees below $\mathbf{0}'$ which depends only on the triple jumps of the $\mathbf{x}_i$ is definable in $\mathscr{D}(\leq \mathbf{0}')$ iff it is arithmetic. The proof however requires defining isomorphisms between different models of arithmetic coded below $\mathbf{0}'$. The only approach that we see

to achieving this, however, involves improving some of the result of Slaman and Woodin [**1986**] (which we hope to do in another paper) and proving some new (but not surprising) results about $\mathscr{D}(\leq \mathbf{0}')$ which would also take us too far afield.

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