THE SPACE OF MINIMAL PRIME IDEALS OF $C(X)$ NEED NOT BE BASICALLY DISCONNECTED
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ABSTRACT. Problems posed twenty and twenty-five years ago by M. Henriksen and M. Jerison are solved by showing that the space of minimal prime ideals of the ring $C(X)$ of continuous real-valued functions on a compact (Hausdorff) space need not be basically disconnected—or even an $F$-space.

If $R$ is a commutative ring, let $\text{Spec}(R) = S(R)$ denote the set of prime ideals of $R$, $\text{Minspec}(R) = m(R)$ the subset of minimal elements of $S(R)$, and if $R$ has an identity element $\text{Maxspec}(R) = M(R)$, the set of maximal elements of $S(R)$. We impose the hull kernel or Zariski topology on $S(R)$; that is the topology with base $\{h^c(a): a \in R\}$, where $h^c(a) = \{P \in S(R): a \notin P\}$, and we regard $m(R)$ and $M(R)$ as subspaces of $S(R)$. For background, see $[Ho, HJ, and K]$.

Below, $X$ will always denote a Tychonoff space. We are concerned particularly with $m(R)$ in case $R = C(X)$, the ring of all real-valued, continuous functions on $X$, and our aim is to present a solution to a problem posed by M. Henriksen and M. Jerison about $m(C(X))$ in 1961 and 1965; see $[HJ]$. Terms not defined explicitly below may be found in $[GJ]$.

For $f \in C(X)$, let $Z(f) = \{x: f(x) = 0\}$, $\text{coz}(f) = X - Z(f)$, $\text{spt}(f) = \text{Cl}(\text{coz}(f))$, and zero (cozero) sets [supports] are sets of the form $Z(f)$ (coz(f)) [spt(f)] for some $f \in C(X)$. If $x \in X$, $M_x = \{f \in C(X): x \in Z(f)\}$ and $O_x = \{f \in C(X): x \in \text{Int}(Z(f))\}$. $X$ is called an $F$-space if each $P \in S(C(X))$ contains a unique $P' \in m(C(X))$. It is well known (see $[GJ]$) that $M(C(X)) = \{M_x: x \in X\}$ for arbitrary compact Hausdorff $X$, and that $m(C(X)) = \{O_x: x \in X\}$ if $X$ is also an $F$-space.

A pair $A, B$ of subsets of $X$ are said to be completely separated if there is an $f \in C(X)$ such that $f[A] = \{0\}$ and $f[B] = \{1\}$. Completely separated subsets have disjoint closures, and the converse holds if $X$ is a normal space.

$X$ is basically (extremally) disconnected if whenever $A$ is an open set and $B$ a cozero (open) set disjoint from $A$, then $A$ and $B$ are completely separated. As shown in $[GJ]$, $X$ is basically (extremally) disconnected iff the closure of each cozero set (open set) is open. Thus every extremally disconnected space is basically disconnected. It is also shown in $[GJ]$ that $X$ is an $F$-space iff disjoint cozero sets are completely separated, so basically disconnected spaces are all $F$-spaces. $X$ is called an $F'$-space if disjoint cozero sets have disjoint closures. Every $F$-space is an $F'$-space but the converse may fail if $X$ is nonnormal $[GH]$.

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As usual we let $\beta X$ denote the Stone-Cech compactification of $X$ and $X^*$ denote $\beta X - X$. We let $N$ denote the countably infinite discrete space. The following are established in [HJ]:

(a) $m(C(X))$ is countably compact (we have shown in [DHKV] that the closure of each weakly Lindelöf subset is compact),
(b) $m(C(X))$ is basically disconnected if it is locally compact,
(c) $m(C(N^*))$ is nowhere locally compact,
(d) it follows easily from [HJJ], 4.2 that the closure of a countable union of basic open sets (i.e., sets of the form $hc(a)$) is a basic open set.

M. Henriksen and M. Jerison asked in 1961 if $m(G(N^*))$ is basically disconnected, and in 1965 they asked whether for any $A$, $m(G(A))$ fails to be basically disconnected; see [HJ]. By condition (d) these seem natural questions. Below we answer the second question in the affirmative (in ZFC), and the first in the negative if Martin’s Axiom (MA) holds; in particular, if the continuum hypothesis (CH) holds. For a statement of Martin’s Axiom see [Ku].

A key part of the solution is the notion of a $P$-set: it is a nonempty compact subset $K$ of $A$ for which every countable intersection of neighborhoods of $K$ is a neighborhood of $K$. Equivalently, $K$ is a $P$-set iff it is nonempty, compact and completely separated from any cozero set disjoint from it. Clearly if a zero set is a $P$-set, then it is open. Many facts about $P$-sets are given in [vM].

A continuous surjection $g: Y \to Z$ is called irreducible or said to map $Y$ irreducibly onto $Z$ if no proper closed subset of $Y$ is mapped onto $Z$ by $g$. For every compact $Y$, there is a unique extremally disconnected space $EY$ minimal with respect to having an irreducible map onto $X$, which we call the absolute of the space $Y$. It is the Stone space of the Boolean algebra of regular closed subsets of $A$; for background see [Wo].

The proof of the following folk-lemma is an exercise.

1. **Lemma.** If $g: Y \to Z$ is closed and irreducible and $D$ is dense in $Z$ then $f^{-1}[D]$ is dense in $Y$.

As noted above, if $X$ is an $F$-space then $m(C(X)) = \{O_x : x \in X\}$. Thus $x \to O_x$ naturally identifies $X$ with $m(C(X))$.

2. **Lemma.** If $X$ is a compact $F$-space then the map $x \to O_x$ is a homeomorphism from $X$ with the topology whose base is $\{spt(f) : f \in G(A)\}$ to $m(C(X))$ with the hull-kernel topology.

**Proof.** For $f \in C(X)$, $hc(f) = \{O_x : f \notin O_x\} = \{O_x : x \notin Int(Z(f))\} = \{O_x : x \in spt(f)\}$, and as noted above, $\{hc(f) : f \in C(X)\}$ is a base for the hull-kernel topology on $m(C(X))$.

From this characterization of the topology of $m(C(X))$ when $X$ is a compact $F$-space we obtain:

3. **Theorem.** If $X$ is a compact $F$-space in which every zero set is regular closed, and if $X$ contains a $P$-set $E$ which maps irreducibly onto $[0,1]$, then $m(C(X))$ is not an $F'$-space. In particular, $m(C(X))$ is not basically disconnected.

**Proof.** Suppose $p: E \to [0,1]$ is irreducible. Let $A = \{a_n : n \in N\}$, $B = \{b_n : n \in N\}$ be two disjoint, countably infinite dense subsets of $[0,1]$. By Lemma
1, \( p^{-1}[A] \) and \( p^{-1}[B] \) are disjoint dense subsets of \( E \). Since \( E \) is compact, \( p \) has a continuous extension \( g: X \to [0,1] \). For each positive integer \( n \), both \( g^{-1}(a_n) \) and \( g^{-1}(b_n) \) are zero sets and by assumption, each has dense interior. By Lemma 2, \( \text{Int}(g^{-1}(a_n)) \) is open, as well as closed, in the hull-kernel topology, so \( L(A) = \bigcup \{ \text{Int}(g^{-1}(a_n)) : n \in \mathbb{N} \} \), \( L(B) = \bigcup \{ \text{Int}(g^{-1}(b_n)) : n \in \mathbb{N} \} \), are nonempty cozero sets of \( m(C(X)) \), which are clearly disjoint. Our theorem will be proved once we establish that \( E \) is contained in the hull-kernel closure of both \( L(A) \) and \( L(B) \).

Suppose \( x \in E \). By Lemma 2, each basic neighborhood of \( x \) takes the form \( C(X)(U) \), where \( U = \text{coz}(h) \) for some \( h \in C(X) \). Since \( E \) is a P-set and \( C(X)(U) \) meets \( E \), we know that \( U \) meets \( E \). Since \( p^{-1}[A] \) and \( p^{-1}[B] \) are dense in \( E \), \( U \) meets each of them and hence meets both \( g^{-1}[A] \) and \( g^{-1}[B] \). Since \( g^{-1}[A] \) and \( g^{-1}[B] \) are unions of regular closed zero sets, it follows that \( U \) is not disjoint from either \( L(A) \) or \( L(B) \). Thus \( x \) is in the closure of both \( L(A) \) and \( L(B) \), so \( m(C(X)) \) is not an \( F' \)-space.

K. Kunen has shown that if \( MA \) holds then \( N^* \) contains a P-set homeomorphic to the absolute of \( [0,1] \) (see [Ku, Theorem 1.2]). It is well known that \( N^* \) is an \( F \)-space in which every nonempty zero set has nonempty interior [vM, 1.6.2]. Thus the following corollary follows from Theorem 3.

4. COROLLARY (MA). \( m(C(N^*)) \) is not an \( F \)-space.

There is also a compact \( X \) such that \( m(C(X)) \) is not an \( F \)-space whose existence does not depend on \( MA \). To produce it we will need to prove the following.

5. LEMMA. If \( Y \) is a zero set of \( N^* \) with nonempty boundary, then that boundary is a P-set of \( Y \).

PROOF. Let \( Y = Z(f), f \in C(X) \). By duality it will suffice to show that the union of a sequence \( \{ S_n : n \in \mathbb{N} \} \) of closed subsets of \( \text{Int}(Y) \) has closure contained in \( \text{Int}(Y) \). Construct successive clopen subsets \( U_1, \ldots \) such that \( U_1 \) is empty and for each \( n \), \( S_n \cup U_n \subset V_m+1 \subset \text{Int}(Y) \). Next for each \( n \) let \( V_n \) be clopen in \( N^* \) such that \( f^{-1}[[0, 1/(2n+1)]] \subset V_n \subset f^{-1}[[0, 1/2n]] \). Then \( \{ U_n \} \) and \( \{ V_n \} \) are sequences of clopen subsets of \( N^* \) such that \( S_n \subset U_n \subset U_{n+1} \subset V_{m+1} \subset V_m \) and \( \text{Bd}(Y) \subset V_m \) for positive integers \( m, n \). By [W, Chapter 3], there is a clopen subset \( W \) of \( N^* \) such that \( U_n \subset W \subset V_m \) for all \( m, n \); thus

\[
\text{Cl} \left( \bigcup \{ U_n : n \in \mathbb{N} \} \right) \subset W \subset \bigcap \{ V_n : n \in \mathbb{N} \} = Y,
\]

and since \( W \) is open, \( \text{Cl}(\bigcup \{ U_n : n \in \mathbb{N} \}) \subset \text{Int}(Y) \).

Suppose \( S \) and \( T \) are spaces, \( A \) is a closed subspace of \( S \) and \( f: A \to T \) is continuous. Recall that \( S \cup_f T \) is the quotient space of the disjoint union of \( S \) and \( T \) obtained by identifying each \( a \in A \) with \( f(a) \in T \).

6. COROLLARY. There is a zero set \( Z \) of \( N^* \) with a quotient space \( X \) such that \( m(C(X)) \) is not an \( F \)-space.

PROOF. Let \( Z \) be a zero set of \( N^* \) with nonempty boundary. It is well known that the boundary of \( Z \) maps continuously onto \( \beta N \), hence onto \( E[0,1] \). Let \( f \) be such a continuous map of \( Z \) onto \( E[0,1] \). Since \( \text{Bd}(Z) \) is a nowhere dense \( P \)-set of \( Z \) by Lemma 5, the space \( X = Z \cup_f E[0,1] \) is an \( F \)-space in which by [vM, 1.4.1 and 1.4.2] every zero set is a regular closed set. In \( X \) there is a copy of \( E[0,1] \).
7. Remarks and open problems. To apply Theorem 3 to the space \( N^* \) seems to require the existence of a separable infinite \( P \)-set in \( N^* \). As is noted in [vM, problem 6] it is an open problem whether there is such a \( P \)-set in \( N^* \) unless MA or some set-theoretic axiom beyond ZFC holds. Hence the question of whether \( m(C(N^*)) \) is an \( F \)-space remains open in ZFC.

Among other questions which come to mind from the examples given above: Suppose \( X \) is an \( F \)-space in which zero sets are regular closed, and suppose \( X \) fails to contain an infinite \( P \)-set. Must \( m(C(X)) \) be basically disconnected, or even an \( F' \)-space? Is there a compact \( X \) such that \( m(C(X)) \) is basically disconnected but not locally compact? Exactly when is \( m(C(X)) \) basically disconnected?

References


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