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CUT LOCUS CONTAINED IN A HYPERSURFACE

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ABSTRACT. We prove that if the cut locus $C(p)$ of a point p in a compact connected Riemannian manifold M is contained in a connected hypersurface N, then M is homeomorphic to S^m if $C(p) \neq N$ and M is homotopically equivalent to $\mathbf{R}P^m$ if $C(p) = N$.

1. Introduction. Let (M, g) be a compact connected Riemannian manifold, $\dim M = m \geq 2$. Let p be a point in M and $C(p)$ the cut locus of p in M. Suppose that $C(p)$ is contained in N, a codimension one submanifold of M.

The objective of this paper is the proof of the following:

THEOREM. *If $C(p) = N$, then M has the homotopy type of $\mathbf{R}P^m$. If $C(p) \neq N$, then M is homeomorphic to the m-sphere S^m.*

The origin of the problem is the study of the standard situation in $\mathbf{R}P^m$ and S^m. In the real projective space, the cut locus of a point is a hypersurface. In the standard sphere, the cut locus of a point is its antipodal point, which is contained in any meridian hypersurface that contains the two points.

A more general situation is furnished whenever the cut locus is contained in a submanifold. We describe more precisely some particular cases of this situation in the proof below (see Theorem A).

The proof is based on the study of the normal bundle of N in M, its restriction to $C(p)$ and the consideration of the possible trivialization or not of this restriction.

As a general reference for terminology and results on the cut locus we use Besse [**1**]. Actually, we use only topological properties of the cut locus. Thus our theorem could be stated as a result on the topology of compact C^∞ manifolds.

2. Results and proofs. Let (M, g) be a compact connected Riemannian manifold, $\dim M = m \geq 2$, p a point in M and $C(p)$ the cut locus of p in M.

Let N be a connected submanifold of M, $\dim N = n < m$, and suppose that $C(p)$ is contained in N.

Let η be the normal bundle of N in M and η_c the restriction of η to $C(p)$.

THEOREM A. *If η_c is trivial, then M is homeomorphic to the standard m-sphere S^m and $C(p) \neq N$ if $\dim N \neq 0$.*

PROOF. If η_c is trivial, there exists then a diffeomorphism from M onto M sending $C(p)$ to C', disjoint with $C(p)$. Thus M can be covered by two open sets both diffeomorphic to $\mathbf{R}^m$ and hence M must be homeomorphic to S^m by a theorem of Milnor characterizing the sphere [**3**].

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We also have $C(p) \neq N$, because $C(p)$ has the homotopy type of S^m minus one point and so $C(p)$ is contractible. □

THEOREM B. *If $C(p)$ is contained in N, $\dim N = m - 1$ and η_c is not trivial, then M has the homotopy type of $\mathbf{R}P^m$, N has the homotopy type of $\mathbf{R}P^{m-1}$, and $C(p) = N$.*

PROOF. If η_c is not trivial, the sphere bundle associated to η_c is then a double covering $\pi: S \to C(p)$, with S connected. This shows that the fundamental group of $C(p)$ contains a subgroup of index two and, since $C(p)$ is the cut locus of M, the fundamental group of M also contains a subgroup of index two.

Let $\tilde{M} \xrightarrow{\rho} M$ be a double covering of M, with $\tilde{M}$ connected, corresponding to a subgroup of index two of the fundamental group of M. This covering space is trivial on $M - C(p)$ (because $M - C(p)$ is contractible). Therefore we have $\tilde{M} = \tilde{M}_+ \cup \rho^{-1}(C(p)) \cup \tilde{M}_-$, where $\tilde{M}_+$ and $\tilde{M}_-$ are the connected components of $\tilde{M} - \rho^{-1}(C(p))$ and $\rho: \tilde{M}_+ \to M - C(p)$, $\rho: \tilde{M}_- \to M - C(p)$ are diffeomorphisms (we endow $\tilde{M}$ with the obvious C^∞ structure and Riemannian metric induced by ρ).

The normal bundle of $\rho^{-1}(C(p))$ in $\tilde{M}$ (i.e., the restriction to $\rho^{-1}(C(p))$ of the normal bundle $\tilde{\rho}$ of $\rho^{-1}(N)$ in $\tilde{M}$) is trivial since we can choose a normal vector to $\rho^{-1}(C(p))$ at each point which is pointed toward $\tilde{M}_+$, say. This shows that we can easily construct a diffeomorphism from $\tilde{M}$ onto $\tilde{M}$ sending $\rho^{-1}(C(p))$ to a subset K of $\tilde{M}$, such that $\tilde{M} - K$ has two components and one of them contains $\tilde{M}_-$. Thus, there exists an open set M_1 of $\tilde{M}$ which is diffeomorphic to $\mathbf{R}^m$ and contains both $\tilde{M}_-$ and $\rho^{-1}(C(p))$.

Similarly, there exists an open set M_2 of $\tilde{M}$ which is diffeomorphic to $\mathbf{R}^m$ and contains $\tilde{M}_+$ and $\rho^{-1}(C(p))$. Therefore M_1 and M_2 cover $\tilde{M}$, and since both are diffeomorphic to $\mathbf{R}^m$ we conclude by Milnor's theorem [**3**] that $\tilde{M}$ is homeomorphic to S^m. Thus M is the orbit space of the canonical involution on $\tilde{M}$, which clearly does not have fixed points. Therefore M is homotopically equivalent to $\mathbf{R}P^m$ (see [**2**], p. 43).

A Mayer-Vietoris argument shows that $\rho^{-1}(C(p))$ is a homology $(m-1)$-sphere. If $m = 2$, $\tilde{M}$ is homeomorphic to S^2, M is homeomorphic to $\mathbf{R}P^2$ and $C(p)$ has the homotopy type of S^1. Thus $\rho^{-1}(C(p))$ has the homotopy type of S^1. If $m \geq 3$, Seifert–van Kampen's theorem shows that $\rho^{-1}(C(p))$ is simply connected. Thus $\rho^{-1}(C(p))$ is a homotopy $(m-1)$-sphere. Therefore $C(p)$ has the homotopy type of $\mathbf{R}P^{m-1}$ (see [**2**], p. 43).

Observe that $\rho^{-1}(N)$ is connected because $\rho^{-1}(C(p))$ is. Then we have, with $\mathbf{Z}/(2)$ coefficients, the following chain of isomorphisms:

$$\begin{aligned} H_0(\rho^{-1}(N), \rho^{-1}(C(p))) &\approx_{(1)} \overline{H}_c^n(\rho^{-1}(C(p))) \approx_{(2)} \overline{H}^n(\rho^{-1}(C(p))) \\ &\approx_{(3)} \overline{H}^n(S^n) \approx_{(4)} H^n(S^n) \approx \mathbf{Z}/(2) \end{aligned}$$

where $\overline{H}$ (resp. $\overline{H}_c$) denotes Alexander-Spanier cohomology (resp. with compact supports).

(1) is the classical duality isomorphism (see Theorem 10, p. 342 of [**4**]). (2) is an isomorphism because $\rho^{-1}(C(p))$ is compact Hausdorff (see Lemma 11, p. 321 of [**4**]). (3) is an isomorphism because Alexander-Spanier cohomology satisfies the

homotopy axiom. (4) is an isomorphism because Alexander-Spanier coincides with singular cohomology for compact polyhedra.

On the other hand, if $\rho^{-1}(C(p))$ is different from $\rho^{-1}(N)$, then $H_0(\rho^{-1}(N), \rho^{-1}(N)-\rho^{-1}(C(p)))$ will be zero, a contradiction. Therefore $\rho^{-1}(N)=\rho^{-1}(C(p))$, and then $N=C(p)$. This finishes the proof of Theorem B. $\square$

COMMENTS. (1) Theorems A and B prove the result announced at the beginning of the paper.

(2) Theorem A studies the more general case of the cut locus contained in a submanifold of any codimension. Other examples of this situation are the standard complex *o* quaternionic projective spaces, and the Cayley plane.

(3) For N, the condition of being connected is superfluous because $C(p)$ is connected. Moreover, we can assume that N has the maximum possible dimension among those submanifolds containing $C(p)$.

(4) Observe that we have proved that if the cut locus $C(p)$ of a point p in a compact connected Riemannian manifold M is contained in a hypersurface N, then $C(p)=N$ if and only if M is homeomorphic to the m-sphere.

(5) If the cut locus $C(p)$ is a submanifold with codimension bigger than one, and $C(p)$ is not a point, then it is not contained in any hypersurface. This is the situation in the standard $\mathbf{C}P^m$ or $\mathbf{H}P^m$, or in the Cayley plane $\mathrm{Ca}P^2$.

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