AMENABILITY AND DERIVATIONS OF THE FOURIER ALGEBRA

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ABSTRACT. It is shown that a locally compact group $G$ is amenable if and only if every derivation of the Fourier algebra $A(G)$ into a Banach $A(G)$-bimodule is continuous. Also given are necessary and sufficient conditions for $A(G)$ to be weakly amenable.

Introduction. If $G$ is an abelian locally compact group, then it is known that every derivation of $A(G)$, the Fourier algebra of $G$, into a Banach $A(G)$-bimodule is continuous [8, p. 410]. We shall prove in the first part of this paper that this automatic continuity property for derivations of $A(G)$ characterizes amenable locally compact groups.

In the second part of this paper we will examine the derivations of $A(G)$ into commutative Banach $A(G)$-bimodules. We shall focus our attention on the nature of $VN(G)$, the dual of $A(G)$, as a commutative Banach $A(G)$-bimodule and, in particular, on the $A(G)$-submodule $UCB(^{\ast}G)$ of $VN(G)$.

$UCB(^{\ast}G)$ is the norm closure in $VN(G)$ of the linear span of $A(G) \cdot VN(G)$. The definition of $UCB(^{\ast}G)$ is due to E. Granirer, who studied its properties in [5]. Recently, Bade, Curtis, and Dales [1] introduced the notion of a weakly amenable Banach algebra $\mathcal{A}$ as one in which every continuous derivation of $\mathcal{A}$ into a commutative Banach $\mathcal{A}$-bimodule is zero. We will prove that $A(G)$ is weakly amenable if and only if every continuous derivation of $A(G)$ into $UCB(^{\ast}G)$ is zero. We also present evidence to suggest that $A(G)$ may well be weakly amenable for a large class of locally compact groups. In particular, if $G$ is discrete, then $A(G)$ is weakly amenable.

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Definitions and notation. Let $G$ be a locally compact group with a fixed left Haar measure $dx$. For each complex-valued function $f$ on $G$ and every $x \in G$ define $L_x f(y) = f(x^{-1}y)$, $f^\vee(x) = f(x^{-1})$, and $\hat{f}(x) = \overline{f(x^{-1})}$.

$G$ is said to be amenable if there exists $m \in L^\infty(G)^* \setminus \{0\}$ such that $m \geq 0$, $m(1_G) = 1$ and $m(L_x f) = m(f)$ for every $x \in G$, $f \in L^\infty(G)$.

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The Fourier algebra of $G$, denoted by $A(G)$, is the linear subspace of $C_0(G)$ (the continuous complex-valued functions on $G$ which vanish at infinity) consisting of all functions $(f * g)\Gamma$, where $f, g \in L^2(G)$. $VN(G)$ is the closure in the weak operator topology of the linear span of $\{L_x; x \in G\}$ in $B(L^2(G))$, the algebra of bounded linear operators on $L^2(G)$. $A(G)$ is the unique predual of the von Neumann algebra $VN(G)$ [3, pp. 210 and 218]. With pointwise multiplication and $\|u\|_{A(G)} = \sup\{|(T,u)|; T \in VN(G), \|T\| \leq 1\}$, $A(G)$ is a Banach algebra with spectrum $\Delta(A(G)) = G$ [3, p. 222].

If $I$ is an ideal of $A(G)$, let $Z(I) = \{x \in G; u(x) = 0 \text{ for every } u \in I\}$. If $A \subset G$ is closed, then $I(A) = \{u \in A(G); u(x) = 0 \text{ for every } x \in A\}$ is a closed ideal of $A(G)$. $A$ is called a set of spectral synthesis (or simply an $S$-set) if the only closed ideal $I$ in $A(G)$ with $Z(I) = A$ is $I(A)$. An ideal $I$ is called cofinite if $\text{codim} I = \dim(A(G)/I) < \infty$.

An algebraic bimodule $X$ of the Banach algebra $A$ is called a Banach $A$ bimodule if $X$ is a Banach space, $\|u \cdot x\| \leq \|u\| \|x\|$ and $\|x \cdot u\| \leq \|u\| \|x\|$ for every $u \in A$, $x \in X$. $X$ is commutative if $x \cdot u = u \cdot x$ for every $u \in A$, $x \in X$.

$VN(G)$ becomes a commutative Banach $A(G)$-bimodule by means of the formula $(u \cdot T,v) = (T \cdot u,v) = (T,uv)$ for every $T \in VN(G)$, $u,v \in A(G)$. If $T \in VN(G)$, then $\text{supp}T = \{x \in G; u(x) = 0 \text{ for every } u \in A(G) \text{ with } u \cdot T = 0\}$. A map $\Gamma: VN(G) \rightarrow VN(G)$ is said to be invariant if $u \cdot \Gamma(T) = \Gamma(u \cdot T)$ for every $u \in A(G)$, $T \in VN(G)$.

$P(G)$ denotes the continuous positive definite functions on $G$.

**Continuity of derivations on $A(G)$**.

**Definition 1.** Let $A$ be a Banach algebra and let $X$ be a Banach $A$-bimodule. A derivation $D: A \rightarrow X$ is a linear map which satisfies

$$D(uv) = u \cdot D(v) + D(u) \cdot v \quad \text{for every } u,v \in A.$$ 

**Lemma 1.** Let $G$ be a nonamenable locally compact group. Then there exists a discontinuous derivation of $A(G)$ into a finite-dimensional commutative Banach $A(G)$-bimodule.

**Proof.** If $G$ is nonamenable, then $I^2(\{e\}) = \{\sum_{i=1}^n u_i v_i; u_i, v_i \in I(\{e\})\}$ is not closed in $A(G)$ [4, Lemma 6.7]. Therefore, $I^2(\{e\})$ is not cofinite or $I^2(\{e\})$ is cofinite but not closed. In either case, there exists a nonclosed cofinite ideal $I$ of $A(G)$ such that $I^2(\{e\}) \subset I \subset I(\{e\})$. Then, as is shown in [2, p. 402], the finite-dimensional space $I(\{e\})/I$ can be made into a commutative Banach $A(G)$-bimodule and a discontinuous derivation $D: A(G) \rightarrow I(\{e\})/I$ can be constructed. \hfill $\square$

**Lemma 2.** Let $G$ be an amenable locally compact group. Let $I$ be a closed ideal in $A(G)$ with infinite codimension. Then there exist sequences $\{u_n\}$, $\{v_n\}$ in $A(G)$ such that $u_nv_1 \cdots v_{n-1} \notin I$ but $u_nv_1 \cdots v_n \in I$ for all $n \geq 2$.

**Proof.** Let $A = Z(I)$. Since $G$ is amenable and $I$ has infinite codimension, $A$ is infinite [4, Corollary 5.6].

Assume that $x$ is a cluster point of $A$. Let $x_1 \in A$, $x_1 \neq x$. Let $V_1$ be a compact neighborhood of $x_1$ with $x \notin V_1$. Choose $x_2 \in A \setminus V_1$, $x_2 \neq x$. Let $V_2$ be a compact neighborhood of $x_2$ such that $x \notin V_2$ and $V_1 \cap V_2 = \emptyset$. Proceeding inductively,
we get \( x_n \in A \setminus \bigcup_{i=1}^{n-1} V_i, x_n \neq x \) and a compact neighborhood \( V_n \) of \( x_n \) such that \( x \notin V_n \) and \( V_n \cap \bigcup_{i=1}^{n-1} V_i = \emptyset \).

If \( A \) has no cluster points, then there exists a discrete set \( \{x_1, x_2, \ldots \} \subseteq A \). Let \( V_1 \) be a compact neighborhood of \( x_1 \) with \( \{x_2, x_3, \ldots \} \cap V_1 = \emptyset \). With \( V_{n-1} \) chosen, let \( V_n \) be a compact neighborhood of \( x_n \) such that \( V_n \cap \{x_{n+1}, x_{n+2}, \ldots \} = \emptyset \) and \( V_n \cap \bigcup_{i=1}^{n-1} V_i = \emptyset \).

In either case, we get a sequence \( \{V_n\} \) of compact neighborhoods of points \( \{x_n\} \) in \( A \) such that \( V_n \cap \bigcup_{i=1}^{n-1} V_i = \emptyset \) for \( n \geq 2 \).

For each \( n = 1, 2, \ldots \), let \( u_n \) be a function \( A(G) \) with \( u_n \geq 0 \), supp \( u_n \subseteq V_n \), \( u_n(x_n) > 0 \) and \( \|u_n\|_{A(G)} = 1/2^n \).

For \( k = 1, 2, \ldots \) define
\[
 v_k = \sum_{i=k+1}^{\infty} u_i.
\]

Since \( \sum_{k=1}^{\infty} \|u_k\|_{A(G)} < \infty \), each \( v_k \in A(G) \).

If \( n \geq 2 \), then
\[
 u_n v_1 \cdots v_{n-1}(x_n) > 0.
\]
Hence \( u_n v_1 \cdots v_{n-1} \notin I \). However, supp \( u_n \subseteq V_n \) and \( v_n(V_n) = 0 \). Therefore,
\[
 u_n v_1 \cdots v_n = 0 \in I. \quad \square
\]

**THEOREM 1.** Let \( G \) be a locally compact group. Then the following are equivalent:

(i) \( G \) is amenable.

(ii) Every derivation of \( A(G) \) into a finite-dimensional commutative Banach \( A(G) \)-bimodule is continuous.

(iii) Every derivation of \( A(G) \) into a Banach \( A(G) \)-bimodule is continuous.

**PROOF.** That (iii) implies (ii) is trivial. That (ii) implies (i) follows immediately from Lemma 1.

Assume that \( G \) is amenable. Let \( I \) be a closed cofinite in \( A(G) \). By [4, Corollary 6.6], \( I^2 = I \). This, together with Lemma 2 and [6, Theorem 2], implies that every derivation from \( A(G) \) into a Banach \( A(G) \)-bimodule is continuous. That is, (i) implies (iii). \( \square \)

**Weak amenability of \( A(G) \).**

**DEFINITION 2.** Let \( \varphi \) be a multiplicative linear functional on a Banach algebra \( \mathcal{A} \). A point derivation at \( \varphi \) is a linear function \( d: \mathcal{A} \rightarrow \mathbb{C} \) such that
\[
 d(uv) = \varphi(u) d(v) + \varphi(v) d(u) \quad \text{for every} \ u, v \in \mathcal{A}.
\]

**PROPOSITION 1.** Let \( G \) be a locally compact group. Then \( A(G) \) has no continuous nonzero point derivations at any point in the spectrum of \( A(G) \).

**PROOF.** Let \( x \in G = \Delta(A(G)); \) cf. [3, p. 229]. Let \( d \) be a continuous point derivation at \( x \). Let \( v \in A(G) \) be such that \( v(x) = 1 \) and \( \|v\|_{A(G)} = 1 \). Then
\[
 d(v^n) = nd(v) \quad \text{for} \ n = 1, 2, \ldots.
\]
Since \( d \) is bounded, \( d(v) = 0 \). Let \( v_1, v_2 \in I(\{x\}) \). Then
\[
 d(v_1 v_2) = v_1(x) d(v_2) + v_2(x) d(v_1) = 0.
\]
$I^2(\{x\})$ is an ideal in $A(G)$ with $Z(I^2(\{x\})) = \{x\}$. As $\{x\}$ is an $S$-set [3, p. 229], $I^2(\{x\})$ is dense in $I(\{x\})$. Therefore

$$d(u) = 0 \text{ for every } u \in I(\{x\}).$$

However, if $u \in A(G)$, then $u = u(x)v + (u - u(x)v)$ and

$$d(u) = u(x)d(v) + d(u - u(x)v) = 0. \quad \Box$$

**Lemma 3.** Let $G$ be a locally compact group. Let $x \in G$. Then there exists a continuous invariant projection of $VN(G)$ onto $\langle L_x \rangle = \{\lambda L_x; \lambda \in \mathbb{C}\}$.

**Proof.** We follow an idea of P. Renaud [7]. Let $\{V_\alpha\}_{\alpha \in A}$ be a neighborhood basis at $e$. Let $u_\alpha \in P(G) \cap A(G)$ be such that $u_\alpha(e) = 1$ and $\text{supp} u_\alpha \subseteq V_\alpha$. Let $m \in VN(G)^*$ be a weak-* cluster point of $\{L_x u_\alpha\}$. Then, as in [7, Proposition 3 and Theorem 4],

$$m(u \cdot T) = u(x)m(T) \quad \text{for every } u \in A(G), T \in VN(G).$$

Furthermore, $m(L_x) = 1$. Therefore, define

$$P(T) = m(T)L_x.$$ 

$P$ is an invariant projection of $VN(G)$ onto $\langle L_x \rangle$. \Box

**Lemma 4.** Let $D: A(G) \rightarrow VN(G)$ be a continuous derivation. Let $X \in G$ and let $P_x$ be a continuous invariant projection of $VN(G)$ onto $\langle L_x \rangle$. Define

$$D_{P_x}(u) = P_x(D(u)) \quad \text{for every } u \in A(G).$$

Then $D_{P_x}$ is a continuous point derivation at $x$. In particular, $D_{P_x} = 0$.

**Proof.** Let $u, v \in A(G)$. Then

$$D_{P_x}(uv) = P_x(D(uv)) = P_x(u \cdot D(v) + v \cdot D(u))$$

$$= u \cdot P_x(D(v)) + v \cdot P_x(D(u))$$

$$= u(x)D_{P_x}(v) + v(x)D_{P_x}(u). \quad \Box$$

Let $X$ be a commutative Banach $A(G)$-bimodule and let $D: A(G) \rightarrow X$ be a continuous derivation. For each $\varphi \in X^*$ and $u \in A(G)$ define $T_{\varphi,u} \in VN(G)$ by

$$\langle T_{\varphi,u}, V \rangle = \varphi(v \cdot D(u)) \quad \text{for every } v \in A(G).$$

The main idea in the next lemma is due to Bade, Curtis, and Dales [1].

**Lemma 5.** Let $X$ be a commutative Banach $A(G)$-bimodule and let $D: A(G) \rightarrow X$ be a continuous derivation. For each $\varphi \in X^*$ and $u \in A(G)$, define $\tilde{D}_\varphi(u) = T_{\varphi,u}$. Then $\tilde{D}_\varphi: A(G) \rightarrow VN(G)$ is a continuous derivation. Furthermore, if $D$ is nonzero, then for some $\varphi \in X^*$, $\tilde{D}_\varphi$ is nonzero.

**Proof.** Let $u, v, w \in A(G)$.

$$(\tilde{D}_\varphi(uv))(w) = \varphi(w \cdot D(uv)) = \varphi(w \cdot u \cdot D(v) + w \cdot v \cdot D(u))$$

$$= (\tilde{D}_\varphi(v))(uw) + (\tilde{D}_\varphi(u))(vw)$$

$$= [u \cdot \tilde{D}_\varphi(v)](w) + [v \cdot \tilde{D}_\varphi(u)](w).$$

Hence $\tilde{D}_\varphi$ is a derivation. Since $\varphi \in X^*$ and $D$ is continuous, $\tilde{D}_\varphi$ is also continuous.
As \( A(G)^2 \) is dense in \( A(G) \) [3, p. 223], if \( D \) is nonzero, there exist \( u, v \in A(G) \) such that \( D(uv) \neq 0 \). However,
\[
    uv = \frac{(u+v)^2 - u^2 - v^2}{2},
\]
so we may assume that \( D(u^2) \neq 0 \) for some \( u \in A(G) \). Let \( \varphi \in X^* \) be such that \( \varphi(D(u^2)) \neq 0 \). Then
\[
    [\hat{D}\varphi(u)](u) = \varphi(u \cdot D(u)) = \frac{1}{2} \varphi(D(u^2)) \neq 0. \quad \square
\]

**Lemma 6.** Let \( D: A(G) \to VN(G) \) be a continuous derivation. Then \( D(u) \in UCB(\hat{G}) \) for every \( u \in A(G) \).

**Proof.** Let \( u, v \in A(G) \). Then \( D(uv) = uD(v) + vD(u) \in UCB(\hat{G}) \). Since \( A^2(G) \) is dense in \( A(G) \) and \( UCB(\hat{G}) \) is closed, \( D(u) \in UCB(\hat{G}) \) for every \( u \in A(G) \). \( \square \)

**Theorem 2.** Let \( G \) be a locally compact group. Then the following are equivalent:
(i) \( A(G) \) is weakly amenable.
(ii) Every continuous derivation from \( A(G) \) into \( UCB(\hat{G}) \) is zero.

**Proof.** That (i) implies (ii) is trivial. Conversely, it follows from Lemma 5 and Lemma 6 that if \( A(G) \) is not weakly amenable, then there exists a nonzero continuous derivation of \( A(G) \) into \( UCB(\hat{G}) \). \( \square \)

Theorem 2 is a stronger version of [1, Theorem 1.5].

**Proposition 2.** Let \( D: A(G) \to UCB(\hat{G}) \) be a continuous derivation and let \( u \in A(G) \). Then \( \text{supp} \, D(u) \) contains no isolated points.

**Proof.** Assume that \( V \) is an open neighborhood of \( x \in \text{supp} \, D(u) \) with \( V \cap \text{supp} \, D(u) = \{x\} \). Let \( v \in A(G) \) be such that \( v(x) = 1 \) and \( \text{supp} \, v \subseteq V \). Define \( D_0: A(G) \to UCB(\hat{G}) \) by \( D_0(w) = v \cdot D(w) \) for every \( w \in A(G) \). Then \( D_0 \) is a continuous derivation. As \( v(x) = 1, x \in \text{supp} \, D_0(u) \) [3, p. 225]. But \( \text{supp} \, v \cdot D(u) \subseteq \text{supp} \, v \cap \text{supp} \, D(u) = \{x\} \). Therefore \( \text{supp} \, D_0(u) = \{x\} \) and \( D_0(u) = \lambda L_x \) for some \( \lambda \neq 0 \) [3, p. 229].

By lemma 3, there exists a continuous invariant projection \( P_x \) of \( VN(G) \) onto \( \langle L_x \rangle \). By lemma 4, \( P_x \circ D_0 \) is a nonzero continuous point derivation of \( A(G) \) at \( x \), contradicting Proposition 1. \( \square \)

**Theorem 3.** Let \( G \) be a discrete group. Then \( A(G) \) is weakly amenable.

**Proof.** This follows immediately from Theorem 2 and Proposition 2 if we note that \( \text{supp} \, T = \emptyset \) if and only if \( T = 0 \) [3, p. 224]. \( \square \)

**Theorem 4.** Let \( G \) be a discrete group. Then the following are equivalent:
(i) \( G \) is amenable.
(ii) Every derivation from \( A(G) \) into a finite-dimensional commutative Banach \( A(G) \)-bimodule is zero.

**Proof.** If \( G \) is amenable, then every derivation is continuous (Theorem 1). Therefore (ii) holds by Theorem 3. If \( G \) is nonamenable, then by Lemma 1, there
exists a discontinuous derivation of $A(G)$ into a commutative finite-dimensional Banach $A(G)$-bimodule. □

It seems that $A(G)$ is weakly amenable for every locally compact group $G$ (or at least for all amenable locally compact groups). We feel that the key to the proof of this statement lies in a stronger version of Proposition 2.

REFERENCES


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