A NONLINEAR ERGODIC THEOREM FOR A REVERSIBLE SEMIGROUP OF LIPSCHITZIAN MAPPINGS IN A HILBERT SPACE

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ABSTRACT. Let C be a nonempty closed convex subset of a Hilbert space, S a right reversible semitopological semigroup, \( \mathcal{S} = \{ T_t : t \in S \} \) a continuous representation of S as Lipschitzian mappings on a closed convex subset C into C, and \( F(\mathcal{S}) \) the set of common fixed points of mappings \( T_t, t \in S \). Then we deal with the existence of a nonexpansive retraction \( P \) of C onto \( F(\mathcal{S}) \) such that \( PT_t = T_tP = P \) for each \( t \in S \) and \( Px \) is contained in the closure of the convex hull of \( \{ T_tx : t \in S \} \) for each \( x \in C \).

1. Introduction. Let C be a closed convex subset of a real Hilbert space \( H \) and \( T \) be a mapping of C into itself. \( T \) is said to be a Lipschitzian mapping if for each \( n \geq 1 \) there exists a positive real number \( k_n \) such that

\[
\| T^n x - T^n y \| \leq k_n \| x - y \|
\]

for all \( x, y \in C \). A Lipschitzian mapping is said to be nonexpansive if \( k_n = 1 \) for all \( n \geq 1 \) and asymptotically nonexpansive if \( \lim_{n \to \infty} k_n = 1 \), respectively. We denote by \( F(T) \) the set of fixed points of \( T \). The first nonlinear ergodic theorem for nonexpansive mappings was established in 1975 by Bâillon [1]: Let C be a closed convex subset of a Hilbert space and \( T \) be a nonexpansive mapping of C into itself. If the set \( F(T) \) is nonempty, then for each \( x \in C \), the Cesàro means

\[
S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k x
\]

converge weakly to some \( y \in F(T) \). In this case, putting \( y = Px \) for each \( x \in C \), \( P \) is a nonexpansive retraction of C onto \( F(T) \) such that \( PT = TP = P \) and \( Px \in \overline{\{ T^n x : n = 0, 1, 2, \ldots \}} \) for each \( x \in C \), where \( \overline{A} \) is the closure of the convex hull of \( A \). In [11 and 12], Takahashi proved the existence of such a retraction—"ergodic retraction"—for an amenable and a right reversible semigroup of nonexpansive mappings in a Hilbert space. On the other hand, Hirano and Takahashi [4] proved the Cesàro means for asymptotically nonexpansive mappings converge weakly to a fixed point.

In this paper, we deal with the existence of "ergodic retraction" for a right reversible semigroup of Lipschitzian mappings; that is, we prove a nonlinear ergodic theorem for such a semigroup in a Hilbert space. This theorem is a generalization...
of the result [4, 12]. Furthermore, we prove a weak convergence theorem which is similar to that of [7, 9].

2. Nonlinear ergodic theorem. Let $S$ be a semitopological semigroup, i.e., $S$ is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \to a \cdot s$ and $s \to s \cdot a$ from $S$ to $S$ are continuous. Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. Then a family $\mathcal{T} = \{T_t: t \in S\}$ of mappings from $C$ into itself is said to be a Lipschitzian semigroup on $C$ if $\mathcal{T}$ satisfies the following:

1. $T_{ts}(x) = T_t T_s(x)$ for $t, s \in S$ and $x \in C$;
2. the mapping $(s, x) \to T_s(x)$ from $S \times C$ into $C$ is continuous when $S \times C$ has the product topology;
3. for each $s \in S$, there exists $k_s > 0$ such that $||T_s(x) - T_s(y)|| \leq k_s ||x - y||$ for $x, y \in C$.

A semitopological semigroup $S$ is right reversible if any two closed left ideals of $S$ have nonvoid intersection. In this case, $(S, \leq)$ is a directed system when the binary relation “$\leq$” on $S$ is defined by $a \leq b$ if and only if $\{a\} \cup S a \supseteq \{b\} \cup S b$. Let $F(\mathcal{T})$ denote the set $\{x \in C: T_s x = x$ for all $s \in S\}$ of common fixed points of mappings $T_s$, $s \in S$ in $C$ (fixed point theorems for Lipschitzian semigroups were proved in [6, 3]). Then we have the following:

**THEOREM 1.** Let $C$ be a closed convex subset of a real Hilbert space $H$ and let $S$ be a right reversible semitopological semigroup. Let $\mathcal{T} = \{T_t: t \in S\}$ be a Lipschitzian semigroup on $C$ with $\limsup_{t} k_t \leq 1$. Then $F(\mathcal{T})$ is a closed convex subset of $C$.

**PROOF.** Closedness of $F(\mathcal{T})$ is obvious. To show convexity it is sufficient to prove that $z = (x + y)/2 \in F(\mathcal{T})$ for all $x, y \in F(\mathcal{T})$. We have

$$||T_t z - x||^2 = ||T_t z - T_t x||^2 \leq k_t^2 ||z - x||^2 = \frac{1}{4} k_t^2 ||x - y||^2$$

and

$$||T_t z - y||^2 = ||T_t z - T_t y||^2 \leq k_t^2 ||z - y||^2 = \frac{1}{4} k_t^2 ||x - y||^2.$$ 

Thus

$$||T_t z - z||^2 = \frac{1}{2} ||T_t z - x||^2 + \frac{1}{2} ||T_t z - y||^2 - \frac{1}{4} ||x - y||^2 \leq \frac{1}{4} (k_t^2 - 1) ||x - y||^2$$

and hence $\lim_t ||T_t z - z|| = 0$. Therefore we obtain

$$z = \lim_t T_t z = \lim_t T_t z = \lim_t T_s T_t z = T_s \lim_t T_t z = T_s z$$

for all $s \in S$.

Let $\{x_\alpha: \alpha \in A\}$ be a bounded net of a Hilbert space $H$ and let $C$ be a closed convex subset of $H$. Then we define

$$r(x) = \limsup_{\beta \to \alpha} ||x_\alpha - x|| \quad \text{and} \quad \rho = \inf \{r(x): x \in C\}.$$

It is well known that there exists a unique element $a \in C$ with $r(a) = \rho$, called the asymptotic center of $\{x_\alpha: \alpha \in A\}$ in $C$. A useful lemma is a result proved in [5], which we state here as:

**LEMMA 1.** $r^2 + ||a - x||^2 \leq r(x)^2$ for every $x \in C$.

As a consequence, we have the following.
LEMMA 2 (LIM [8]). Let \( \{y_\beta\} \) be a net of \( C \) such that \( \limsup_\beta r(y_\beta) \leq r \). Then \( y_\beta \rightarrow a \).

PROOF. By Lemma 1, we have \( r^2 + \|a - y_\beta\|^2 \leq r(y_\beta)^2 \) and hence
\[
r^2 + \limsup_\beta \|a - y_\beta\|^2 \leq \limsup_\beta r(y_\beta)^2 \leq r^2.
\]
Therefore we have \( \limsup_\beta \|a - y_\beta\|^2 = 0 \) and this implies \( y_\beta \rightarrow a \).

We also know that if \( \{x_\alpha\} \subseteq C \) and if \( \{x_\alpha\} \) converges weakly to \( y \in C \) then \( y = a \) [2, Theorem 4.2]. Let \( Q \) be the metric projection of \( H \) onto \( F(\mathcal{S}) \). Then, by Phelps [10], \( Q \) is nonexpansive. Before proving a nonlinear ergodic theorem, we prove the following crucial lemma.

LEMMA 3. Let \( C \) be a closed convex subset of a real Hilbert space \( H \) and let \( S \) be a right reversible semitopological semigroup. Let \( \mathcal{S} = \{T_t: t \in S\} \) be a Lipschitzian semigroup on \( C \) with \( \limsup_s k_s \leq 1 \). Suppose that \( F(\mathcal{S}) \neq \emptyset \). Then for each \( x \in C \), \( \{QT_s x: s \in S\} \) converges to the asymptotic center of \( \{T_s x: s \in S\} \) in \( F(\mathcal{S}) \).

PROOF. Let \( z \) be the asymptotic center of \( \{T_s x: s \in S\} \) in \( F(\mathcal{S}) \). Then, for all \( s, t \in S \) we obtain
\[
r(QT_s x) \leq \sup_{a \geq t} ||T_a x - QT_s x|| = \sup_{a \geq t} ||T_a x - QT_s x||
\]
\[
= \sup_{a \geq t} ||T_a T_s x - T_a QT_s x|| \leq \left( \sup_{a \geq t} k_a \right) ||T_a x - QT_s x||
\]
\[
\leq \left( \sup_{a \geq t} k_a \right) ||T_s x - z||
\]
and hence
\[
r(QT_s x) \leq \limsup_t k_t ||T_s x - z|| \leq ||T_s x - z||
\]
for all \( s \in S \). Therefore we have \( \limsup_s r(QT_s x) \leq \limsup_s ||T_s x - z|| \leq r \). By Lemma 2, we obtain \( QT_s x \rightarrow z \).

THEOREM 2. Let \( C \) be a closed convex subset of a Hilbert space \( H \) and let \( S \) be a right reversible semitopological semigroup. Let \( \mathcal{S} = \{T_t: t \in S\} \) be a Lipschitzian semigroup on \( C \) with \( \limsup_s k_s \leq 1 \). Suppose that
\[
F(\mathcal{S}) = \bigcap \{F(T_s): s \in S\} \neq \emptyset.
\]
Then the following are equivalent:
(a) \( \bigcap_{s \in S} \overline{co}\{T_t x: t \geq t\} \cap F(\mathcal{S}) \neq \emptyset \) for each \( x \in C \).
(b) There is a nonexpansive retraction \( P \) of \( C \) onto \( F(\mathcal{S}) \) such that \( PT_t = T_t P = P \) for every \( t \in S \) and \( Px \in \overline{co}\{T_t x: t \in S\} \) for every \( x \in C \).

PROOF. (b)⇒(a). Let \( x \in C \). Then \( Px \in F(\mathcal{S}) \). Also
\[
Px \in \bigcap_{s \in S} \overline{co}\{T_t x: t \geq s\}.
\]
In fact,
\[
Px = PT_s x \in \overline{co}\{T_t T_s x: t \in S\} \subseteq \overline{co}\{T_t x: t \geq s\}
\]
for every \( s \in S \).
(a) ⇒ (b). Let \( x \in C \) and \( f \in F(\mathcal{S}) \). Then for each \( s, t \in S \), we have
\[
\limsup_{a} ||T_{a}x - f|| \leq \sup_{a \geq ts} ||T_{a}x - f|| = \sup_{a \geq t} ||T_{as}x - f||
\]
\[
= \sup_{a \geq t} ||T_{as}x - T_{a}f|| \leq \left( \sup_{a \geq t} k_{a} \right) ||T_{s}x - f||
\]
and hence
\[
\limsup_{a} ||T_{a}x - f|| \leq \left( \limsup_{t} k_{t} \right) ||T_{s}x - f|| \leq ||T_{s}x - f||
\]
for every \( s \in S \). So \( \limsup_{s} ||T_{s}x - f|| \leq \liminf_{s} ||T_{s}x - f|| \) and hence the \( \lim_{s} ||T_{s}x - f|| \) exists. Let \( Q \) be the metric projection of \( H \) onto \( F(\mathcal{S}) \). Then by Lemma 3, \( \{QT_{s}x\} \) converges to the asymptotic center \( z \) of \( \{T_{s}x: s \in S\} \) in \( F(\mathcal{S}) \). Let \( u \in \bigcap_{s \in S} \text{co}\{T_{s}x: t \geq s\} \cap F(\mathcal{S}) \). Then, since
\[
||z - u||^{2} = ||T_{s}x - u||^{2} - ||T_{s}x - z||^{2} - 2(z - u, T_{s}x - z)
\]
for every \( s \in S \), we have
\[
||z - u||^{2} + 2\lim_{s}(z - u, T_{s}x - z) = \lim_{s} ||T_{s}x - u||^{2} - \lim_{s} ||T_{s}x - z||^{2}
\]
\[
= \lim_{s} ||T_{s}x - u||^{2} - \limsup_{s} ||T_{s}x - z||^{2}
\]
\[
= r(u)^{2} - r^{2} \geq 0.
\]
Let \( \varepsilon > 0 \). Then we have
\[
2\lim_{s}(z - u, T_{s}x - z) > -||z - u||^{2} - \varepsilon.
\]
Hence there exists \( s_{0} \in S \) such that
\[
2(z - u, T_{s}x - z) > -||z - u||^{2} - \varepsilon
\]
for every \( s \geq s_{0} \). Since \( u \in \text{co}\{T_{t}x: t \geq s_{0}\} \), we have
\[
2(z - u, u - z) \geq -||z - u||^{2} - \varepsilon.
\]
This inequality implies \( ||z - u||^{2} \leq \varepsilon \). Since \( \varepsilon \) is arbitrary, we have \( z = u \). Therefore
\[
\bigcap_{s \in S} \text{co}\{T_{s}x: t \geq s\} \cap F(\mathcal{S}) = \{z\}.
\]
Set \( Px = \lim_{t} QT_{t}x \) for every \( x \in C \). Then we have \( T_{s}Px = Px \) and
\[
PT_{s}x = \lim_{t} QT_{s}T_{t}x = \lim_{t} QT_{ts}x = Px
\]
for every \( s \in S \) and \( x \in C \). From \( \{Px\} = \bigcap_{s \in S} \text{co}\{T_{s}x: t \geq s\} \cap F(\mathcal{S}) \), it is obvious that \( Px \in \text{co}\{T_{s}x: s \in S\} \) for each \( x \in C \). Since
\[
||Px - Py|| = \lim_{t} ||QT_{t}x - QT_{t}y|| \leq \limsup_{t} ||T_{t}x - T_{t}y||
\]
\[
\leq \left( \limsup_{t} k_{t} \right) ||x - y|| \leq ||x - y||
\]
for every \( x, y \in C \), it follows that \( P \) is nonexpansive.

We now turn to consider the weak convergence of \( \{T_{s}x: s \in S\} \) and obtain the results similar to these of Lau [7] and Passty [9]. We denote by \( \omega(x) \) the set of all weak limit points of subnets of the net \( \{T_{s}x: s \in S\} \). Note that if \( F(\mathcal{S}) \) is nonempty then \( \{T_{s}x: s \in S\} \) is bounded and hence \( \omega(x) \) is nonempty. We start with proving the following lemma.
**Lemma 4.** Let $C$ be a closed convex subset of a Hilbert space $H$ and let $S$ be a right reversible semitopological semigroup. Let $\mathcal{S} = \{T_t: t \in S\}$ be a Lipschitzian semigroup on $C$ with $\limsup_{s \to \infty} k_s \leq 1$. Suppose that $F(\mathcal{S}) \neq \emptyset$. Let $x \in C$. If $\omega(x) \subseteq F(\mathcal{S})$, then the net $\{T_s x: s \in S\}$ converges weakly to some $y \in F(\mathcal{S})$.

**Proof.** By Lemma 3, the net $\{QT_s x: s \in S\}$ converges strongly to some $y \in F(\mathcal{S})$. Since $\omega(x) \subseteq \bigcap_{s \in S} \overline{\text{co}}\{T_t x: t \geq s\}$, we have

$$\bigcap_{s \in S} \overline{\text{co}}\{T_t x: t \geq s\} \cap F(\mathcal{S}) \ni \omega(x) \neq \emptyset.$$ 

Hence, as in the proof of Theorem 2, we obtain

$$\{y\} = \bigcap_{s \in S} \overline{\text{co}}\{T_t x: t \geq s\} \cap F(\mathcal{S}).$$

Therefore we have $\{y\} = \omega(x)$.

Let $C$ be a closed convex subset of a Hilbert space $H$ and let $\mathcal{S} = \{T_t: t \in S\}$ be a Lipschitzian semigroup on $C$. A subset $G$ of $S$ is called a uniformly generating set of $\mathcal{S}$ if for each $s \in S$ and $\varepsilon > 0$, there exist $g_1, g_2, \ldots, g_m \in G$ such that

$$\|T_s x - T_{g_1} g_2 \cdots g_m x\| < \varepsilon$$

for every $x \in C$.

**Theorem 3.** Let $C$ be a closed convex subset of a Hilbert space $H$ and let $S$ be a right reversible semitopological semigroup. Let $\mathcal{S} = \{T_t: t \in S\}$ be a Lipschitzian semigroup on $C$ with $\limsup_{s \to \infty} k_s \leq 1$. Suppose that $F(\mathcal{S}) \neq \emptyset$. Let $x \in C$. If $\lim_{s \to \infty} \|T_g x - T_s x\| = 0$ for all $g$ in a uniformly generating set $G$ of $\mathcal{S}$, then the net $\{T_s x: s \in S\}$ converges weakly to some $y \in F(\mathcal{S})$.

**Proof.** By Lemma 4, it suffices to show that $\omega(x) \subseteq F(\mathcal{S})$. Let $\{T_{s_\alpha} x\}$ be a subnet of $\{T_s x: s \in S\}$ converging weakly to some $y \in C$. Let $\varepsilon > 0$. Then for each $t \in S$, there exist $g_1, g_2, \ldots, g_m \in G$ such that

$$\|T_t T_{s_\alpha} x - T_{g_1} g_2 \cdots g_m T_{s_\alpha} x\| < \varepsilon$$

for every $x \in C$. Hence we have

$$\|T_{s_\alpha} x - T_t y\| \leq \|T_{s_\alpha} x - T_{g_1} g_2 \cdots g_m s_\alpha x\| + \|T_{g_1} g_2 \cdots g_m s_\alpha x - T_{s_\alpha} x\| + \|T_{s_\alpha} x - T_t y\|$$

and

$$\limsup_{\alpha} \|T_{s_\alpha} x - T_t y\| \leq \varepsilon + k_t \limsup_{\alpha} \|T_{s_\alpha} x - y\|.$$

Since $\varepsilon$ is arbitrary, we have

$$\limsup_{\alpha} \|T_{s_\alpha} x - T_t y\| \leq k_t \limsup_{\alpha} \|T_{s_\alpha} x - y\|.$$
On the other hand, since $y$ is the asymptotic center of $\{T_{s_t}x\}$ in $C$ we obtain

$$\limsup_t r(T_t y) \leq \left( \limsup_t k_t \right) r(y) \leq r$$

and hence $T_t y \to y$. Therefore we have $y \in F(\mathcal{S})$.

**Remark.** Let $\gamma$ be a positive real number and let $\mathcal{S} = \{T_t : t \in S\}$ be a Lipschitzian semigroup with $\limsup_t k_s \leq \gamma$. Then, putting $k'_s = \sup_{t \geq s} k_t$, we have

$$\|T_s x - T_s y\| \leq k_s \|x - y\| \leq \sup_{t \geq s} k_t \|x - y\| = k'_s \|x - y\|$$

and $\lim_s k'_s = \limsup_s k_s$. Hence $\mathcal{S}$ is a Lipschitzian semigroup with $\lim_s k'_s \leq \gamma$.

**References**


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