PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 2, October 1988

A NEW PROOF OF SONINE'S FORMULA

KRZYSZTOF STEMPAK

(Communicated by George R. Sell)

ABSTRACT. We give a new proof of an old integral formula due to Sonine in which the Bessel functions are involved. By considering the Banach algebra of radial functions on $\mathbf{R}^n$, $n \geq 2$, we observe that Sonine's formula is valid for all positive integers. Next, a complex function theory argument is applied to obtain the validity of the formula for all complex parameters z with $\operatorname{Re} z > 0$.

Among the huge quantity of formulae listed in Watson's classical monograph [**4**] there is the following one (cf. [**4**, p. 411]) due to Sonine:

$$(1)\quad \int_0^\infty J_\nu(at)J_\nu(bt)J_\nu(ct)t^{1-\nu}\,dt = 2^{\nu-1}\Gamma\left(\nu+\frac{1}{2}\right)^{-1}\Gamma\left(\frac{1}{2}\right)^{-1}\frac{\Delta(a,b,c)^{2\nu-1}}{(abc)^\nu}.$$

J_ν denotes here the Bessel function of order ν, $\operatorname{Re}\nu > -\frac{1}{2}$, and $\Delta(a,b,c)$ means the area of a triangle with sides $a,b,c > 0$. If such a triangle does not exist then the integral in (1) equals 0.

The aim of this note is to prove (1) using a group-theoretic argument and a result which is an application of a theorem of Phragmén-Lindelöf type. The idea of using a theorem of such type has been suggested to the author in a conversation with Andrzej Hulanicki to whom I am very indebted.

As a matter of fact, we will apply the following

THEOREM [**3**, p. 186]. *Suppose $f(z)$ is an analytic function on $\operatorname{Re} z > -\delta$, $\delta > 0$, which is of the form $O(e^{k|z|})$ on $\operatorname{Re} z \geq 0$, where $k < \pi$. If $f(z) = 0$ for $z = 0,1,2,\ldots$ then $f = 0$ identically.*

Now, for $x, y \geq 0$ and $\operatorname{Re} z > 0$ denote

$$\phi^{(z)}(t) = (2/t)^{(z-1)/2}\Gamma((z+1)/2)J_{(z-1)/2}(t), \qquad t > 0,$$

and

$$W_{x,y}^{(z)}(t) = 2^{z-2}\Gamma\left(\frac{z+1}{2}\right)\left(\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{1}{2}\right)\right)^{-1}\frac{\Delta(x,y,t)^{z-2}}{(xyt)^{z-1}}t^z,$$
$$|x-y| < t < x+y.$$

In [**2**] it has been shown how an application of a version of the Hankel inversion formula to (1) gives

$$(2)\qquad \phi^{(z)}(ax)\phi^{(z)}(ay) = \int_{|x-y|}^{x+y}\phi^{(z)}(at)W_{x,y}^{(z)}(t)\,dt,$$

Received by the editors October 14, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 33A40; Secondary 33A75.
Key words and phrases. Bessel functions, radial functions.

where $a, x, y \geq 0$ and $\operatorname{Re} z > 0$. In fact the same argument allows us to deduce (1) from (2). Thus, instead of showing (1) we will prove (2). Note, that by homogeneity it suffices to verify (2) for $a = 1$ only.

So, fix $x, y \geq 0$ and consider the functions

$$F(z) = \phi^{(z)}(x)\phi^{(z)}(y) \quad \text{and} \quad G(z) = \int_{|x-y|}^{x+y} \phi^{(z)}(t) W_{x,y}^{(z)}(t)\, dt,$$

which are, clearly, analytic on $\operatorname{Re} z > 0$. We are going to show that $F = G$ identically. To do this remove, first of all, the common term $\Gamma((z+1)/2)^2$ from both F and G. Therefore, instead of F and G we consider

$$\widetilde{F}(z) = \tilde{\phi}^{(z)}(x)\tilde{\phi}^{(z)}(y) \quad \text{and} \quad \widetilde{G}(z) = \int_{|x-y|}^{x+y} \tilde{\phi}_{x,y}^{(z)}(t) \widetilde{W}^{(z)}(t)\, dt,$$

where

$$\tilde{\phi}^{(z)}(t) = \frac{\phi^{(z)}(t)}{\Gamma((z+1)/2)} \quad \text{and} \quad \widetilde{W}_{x,y}^{(z)}(t) = \frac{W_{x,y}^{(z)}(t)}{\Gamma((z+1)/2)}.$$

First, estimate $|\widetilde{F}(z)|$ and $|\widetilde{G}(z)|$. Using the formula

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right)^{-1} \Gamma\left(\frac{1}{2}\right)^{-1} \int_0^\pi e^{it\cos\theta} \sin^{2\nu}\theta\, d\theta,$$

which is valid for $\operatorname{Re} \nu > -\frac{1}{2}$, we easily observe that for $\operatorname{Re} z > 0$,

$$|\tilde{\phi}^{(z)}(t)| \leq \left|\Gamma\left(\frac{z}{2}\right)\right|^{-1} \Gamma\left(\operatorname{Re}\frac{z}{2}\right) \Gamma\left(\operatorname{Re}\frac{z}{2} + \frac{1}{2}\right)^{-1}.$$

Thus, for $\operatorname{Re} z \geq \frac{1}{2}$

$$|\tilde{\phi}^{(z)}(t)| \leq C_0 |\Gamma(z/2)|^{-1}, \qquad C_0 = \max_{s \geq 1/4} \Gamma(s)\Gamma\left(s + \tfrac{1}{2}\right)^{-1}, \tag{3}$$

which implies

$$|\widetilde{F}(z)| \leq C|\Gamma(z/2)|^{-2}, \qquad \operatorname{Re} z \geq 1/2.$$

On the other hand, since

$$\int_{|x-y|}^{x+y} |\widetilde{W}_{x,y}^{(z)}(t)|\, dt = \left|\Gamma\left(\frac{z}{2}\right)\right|^{-1} \Gamma\left(\operatorname{Re}\frac{z}{2}\right) \Gamma\left(\operatorname{Re}\frac{z+1}{2}\right)^{-1} \int_{|x-y|}^{x+y} W_{x,y}^{(\operatorname{Re} z)}(t)\, dt$$

and, cf. e.g. [**2**], for real $s > 0$

$$\int_{|x-y|}^{x+y} W_{x,y}^{(s)}(t)\, dt = 1, \tag{4}$$

then

$$\int_{|x-y|}^{x+y} |\widetilde{W}_{x,y}^{(z)}(t)|\, dt \leq C_0 \left|\Gamma\left(\frac{z}{2}\right)\right|^{-1}, \tag{5}$$

for all $\operatorname{Re} z \geq \frac{1}{2}$. Combining (3) and (5) gives

$$|\widetilde{G}(z)| \leq C \left|\Gamma\left(\frac{z}{2}\right)\right|^{-2}, \qquad \operatorname{Re} z \geq \frac{1}{2}.$$

But, cf. e.g. [**1**, p. 51], for $\operatorname{Re} z \geq \frac{1}{2}$ we have

$$|\Gamma(z)|^{-1} \leq C \exp(\pi |z|/2)$$

and therefore we eventually get

$$|\widetilde{F}(z)| \leq C \exp(\pi|z|/2) \quad \text{and} \quad |\widetilde{G}(z)| \leq C \exp(\pi|z|/2)$$

for $\operatorname{Re} z \geq \frac{1}{2}$.

Now, in the second step, we will show that $\widetilde{F}(z) = \widetilde{G}(z)$, or, what is the same, $F(z) = G(z)$, for $z = 1, 2, 3, \ldots$. To do this consider $L_r^1(\mathbf{R}^n)$, the commutative Banach algebra (under convolution) of radial, integrable functions on the Euclidean group $\mathbf{R}^n$, $n \geq 2$. A function $\tilde{f}$ on $\mathbf{R}^n$ is radial if $\tilde{f}(\bar{x}) = f(||\bar{x}||)$, $\bar{x} \in \mathbf{R}^n$, for a function f on $(0, \infty)$, called the radial part of $\tilde{f}$. Denote $d\mu_n(t) = \sigma(n)t^{n-1}\,dt$ and $L^1(\mu_n) = L^1(\mathbf{R}_+, d\mu_n)$, where $\sigma(n) = 2\pi^{n/2}\Gamma(n/2)^{-1}$ is the volume of the sphere $\Sigma_{n-1} = \{\bar{x} \in \mathbf{R}^n : ||\bar{x}|| = 1\}$.

It is easily seen that radial part of the convolution

$$\tilde{f} * \tilde{g}(\bar{x}) = \int_{\mathbf{R}^n} \tilde{f}(\bar{y}) g(\bar{x} - \bar{y})\, d\bar{y}$$

of two functions $\tilde{f}, \tilde{g} \in L_r^1(\mathbf{R}^n)$ with the radial parts f, g respectively, is the function $f * g$ given by

$$f * g(x) = \int_0^\infty f(y) T^y g(x)\, d\mu_n(y), \qquad x > 0, \tag{6}$$

where the generalized translation T^y, $y \geq 0$, is

$$T^y g(x) = \int_{|x-y|}^{x+y} g(t) W_{x,y}^{(n-1)}(t)\, dt.$$

(We abuse the notation slightly by using the same symbol $*$ to denote the convolution in $\mathbf{R}^n$ and the operation given by (6).) To verify (6) let us introduce the polar coordinates in $\mathbf{R}^n$ letting

$$y_j = y \cos\phi_j \prod_{i=1}^{j-1} \sin\phi_i, \quad j = 1, 2, \ldots, n-1, \qquad y_n = y \prod_{i=1}^{n-1} \sin\phi_i,$$

where $y > 0$ and $0 < \phi_j < \pi$, $j = 1, 2, \ldots, n-2$, $0 < \phi_{n-1} < 2\pi$. The Jacobian of this transformation is

$$\mathscr{J} = y^{n-1} \mathscr{J}_0(\phi_1, \ldots, \phi_{n-1}), \qquad \mathscr{J}_0 = \prod_{j=1}^{n-2} \sin^{n-j-1}\phi_j.$$

Now, for $x = (x, 0, \ldots, 0)$, $\bar{y} = (y_1, \ldots, y_n)$ we get

$$\begin{aligned}
\tilde{f} * \tilde{g}(\bar{x}) &= \int_0^\infty f(y) \int_0^{2\pi} \int_{[0,\pi]^{n-2}} \tilde{g}(x - y\cos\phi_1, \ldots) \\
&\qquad \times \mathscr{J}_0(\phi_1, \ldots, \phi_n)\, d\phi_1 \cdots d\phi_n\, y^{n-1}\, dy \\
&= 2\pi^{(n-1)/2} \Gamma\left(\frac{n-1}{2}\right)^{-1} \int_0^\infty f(y) \int_0^\pi g((x^2 + y^2 - 2xy\cos\phi_1)^{1/2}) \\
&\qquad \times \sin^{n-2}\phi_1\, d\phi_1 y^{n-1}\, dy,
\end{aligned} \tag{7}$$

since

$$\prod_{j=2}^{n-2}\int_0^{\pi}\sin^{n-j-1}\phi_j\,d\phi_j = 2\pi^{(n-1)/2}\Gamma\left(\frac{n-1}{2}\right)^{-1}.$$

Next, by the change of variables $t^2 = x^2 + y^2 - 2xy\cos\phi_1$ we obtain

$$\frac{t\,dt}{xy} = \sin\phi_1\,d\phi_1, \qquad \frac{1}{2}xy\sin\phi_1 = \Delta(x,y,t),$$

so

$$\begin{aligned}(8)\qquad &\int_0^{\pi} g((x^2+y^2-2xy\cos\phi_1)^{1/2})\sin^{n-2}\phi_1\,d\phi_1\\ &= 2^{n-3}\int_{|x-y|}^{x+y} g(t)\frac{\Delta(x,y,t)^{n-3}}{(xy)^{n-2}}t\,dt\\ &= \Gamma\left(\frac{n}{2}\right)^{-1}\Gamma\left(\frac{n-1}{2}\right)\pi^{1/2}\int_{|x-y|}^{x+y} g(t)W_{x,y}^{(n-1)}(t)\,dt.\end{aligned}$$

Combining (7) and (8) gives (6).

It is clear that the correspondence $L_r^1(\mathbf{R}^n) \ni \tilde{f} \mapsto \Phi(\tilde{f}) = f \in L^1(\mu_n)$ establishes the isometric isomorphism of the Banach algebras $L_r^1(\mathbf{R}^n)$ and $L^1(\mu_n)$ where the multiplication in the second one is defined by (6). The fact that $L^1(\mu_n)$, with (6) as the multiplication, forms a commutative Banach algebra may be also verified without any difficulty using only (4) and the following identity:

$$(9)\qquad W_{x,y}^{(n-1)}(t)\,dt\,d\mu_n(y) = W_{x,t}^{(n-1)}(y)\,dy\,d\mu_n(t).$$

Moreover, (9) implies

$$(10)\qquad \int_0^{\infty} T^y g(x)h(x)\,d\mu_n(x) = \int_0^{\infty} g(x)T^y h(x)\,d\mu_n(x),$$

for reasonable functions, e.g. $g \in L^1(\mu_n)$, $h \in L^{\infty}(\mu_n)$.

Now, take $\bar{y} \in \mathbf{R}^n$, $\bar{y} = (1,0,\dots,)$. Since

$$\alpha(\tilde{f}) = \int_{\mathbf{R}^n}\tilde{f}(\bar{x})\exp(i\langle\bar{x},\bar{y}\rangle)\,d\bar{x}$$

defines a multiplicative functional on $L_r^1(\mathbf{R}^n)$, then also $\beta(f) = \alpha(\Phi^{-1}(f))$ gives a multiplicative functional on $L^1(\mu_n)$. But, as one can easily verify

$$\beta(f) = \int_0^{\infty} f(t)\phi^{(n-1)}(t)\,d\mu_n(t)$$

so this and (10) gives

$$\begin{aligned}&\int_0^{\infty}\int_0^{\infty} f(x)g(y)\phi^{(n-1)}(x)\phi^{(n-1)}(y)\,d\mu_n(x)\,d\mu_n(y)\\ &= \int_0^{\infty}\int_0^{\infty} f(x)g(y)T^y\phi^{(n-1)}(x)\,d\mu_n(x)\,d\mu_n(y)\end{aligned}$$

for any $f, g \in L^1(\mu_n)$. By continuity of $\phi^{(n-1)}(t)$ we get

$$\phi^{(n-1)}(x)\phi^{(n-1)}(y) = T^y\phi^{(n-1)}(x)$$

for every $x, y \geq 0$, which proves $F(z) = G(z)$ for $z = n-1$.

Since the assumptions of the Theorem are satisfied, to be precise, for the function $f(z) = \widetilde{F}(z+1) - \widetilde{G}(z+1)$, analytic for $\operatorname{Re} z > -1$, we conclude $\widetilde{F} = \widetilde{G}$ and also $F = G$ on $\operatorname{Re} z > 0$.

REFERENCES

1. B. C. Carlson, *Special functions of applied mathematics*, Academic Press, New York, 1977.
2. D. L. Guy, *Hankel multiplier transformations and weighted p-norms*, Trans. Amer. Math. Soc. **95** (1960), 137–189.
3. E. C. Titchmarsh, *The theory of functions*, Oxford Univ. Press, London, 1939.
4. G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge Univ. Press, 1962.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GEORGIA, ATHENS, GEORGIA 30602

Current address: Institute of Mathematics, University of Wroclaw, Pl. Grunwaldzki 2/4, 50-384 Wroclaw, Poland