OSCILLATORY SOLUTIONS FOR CERTAIN DELAY-DIFFERENTIAL EQUATIONS

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ABSTRACT. The existence of oscillatory solutions for a certain class of scalar first order delay-differential equations is proved. An application to a delay logistic equation arising in certain models for population variation of a single specie in a constant environment with limited resources for growth is considered.

It is known (cf. [1, 2]) that all solutions of the delay logistic equations

\[ N'(t) = N(t)(a - bN(t) - N(t-1)), \quad t > 0, \]

with \( N(t) = N_0(t) > 0, \ -1 \leq t \leq 0, \) \( N_0 \) continuous, and \( a \) and \( b \) positive constants, satisfy \( N(t) \to a/(b+1) \) as \( t \to \infty \) whenever \( b > 1 \). In [3] it was shown that for any \( b > 0 \), there exists \( a(b) > 0 \) and that if \( 0 < a < a(b) \), there exist solutions \( N(t) \) of (1) which do not oscillate about the equilibrium \( N = a/(b + 1) \); in particular, such that, \( N(t) > a/(b + 1) \) for \( t \geq 0 \). It is the purpose of this paper to show that for this same \( a(b) \), if \( a > a(b) \), there exist oscillatory solutions about this equilibrium solution. In case \( b < 1 \), this is known; in fact, a Hopf bifurcation (cf. [1]) shows the existence for certain \( a \) of nonconstant positive periodic solutions. However, if \( b > 1 \), the fact that some solutions of (1) approach \( a/(b + 1) \) in an oscillatory fashion seems to be new.

The above mentioned result for (1) will follow from a result for a more general scalar delay-differential equation of the form

\[ y'(t) = L(y_t) + N(t, y_t), \quad t > 0. \]

Here \( y_t = y(t + \theta), \ -1 \leq \theta \leq 0 \), and we assume

\( (H_1) \) \( L(\phi) \) is continuous and linear on \( C = C([-1, 0], R) \) and \( N(t, \phi) \) is continuous on \( R \times C \) and satisfies

\[ |N(t, \phi)| \leq M(t)||\phi||^2, \quad \phi \in C, \ ||\phi|| \leq B_0, \ t \geq 0; \]

where the norm in \( C \) is defined by \( ||\phi|| = \sup\{||\phi(\theta)||; \ -1 \leq \theta \leq 0\} \), and \( \int_0^\infty M(t) \, dt < \infty; \)

\( (H_2) \) The characteristic equation for

\[ y'(t) = L(y_t) \]

has a pair of simple pure imaginary roots \( \pm i\beta, \beta > 0, \) and all other roots have negative real parts.

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REMARK 1. Under assumption (H2), there exists a nonconstant periodic solution $y^*(t)$ of (3) and positive numbers $\rho^*$ and $B$, $\rho^* < 1$, $B < B_0/2$, such that
\[
\max\{y^*(t) : t \in R\} \geq \rho^*, \quad \min\{y^*(t) : t \in R\} \leq -\rho^*, \\
|y^*(t)| \leq B, \quad t \in R.
\]
This follows from standard theory for solutions of (3); cf., for example, Hale's monograph [4].

DEFINITION. The real-valued function $f(t)$ on $[0, \infty)$ is oscillatory if there exist $t_n \to \infty$ as $n \to \infty$, $t_{n+1} > t_n$, such that $(-1)^n f(t_n) > 0$, $n = 1, 2, \ldots$.

REMARK 2. If $f(t)$ is continuous and oscillatory in this sense, clearly $f$ must have an unbounded sequence of zeros and cannot be identically zero on any half infinite interval $[t_0, \infty)$, $t_0 \geq 0$.

THEOREM 1. If (H1) and (H2) hold, there exists $\delta_0 > 0$ such that for each $\delta$, $0 < \delta < \delta_0$, (2) has an oscillatory solution $y = w(t)$ such that $|w(t)| \leq \delta$, $t \geq 0$.

PROOF. Let $u(t)$ be the fundamental solution for (3); i.e., let $u(t)$ solve (cf. appendix)
\[
u'(t) = L(ut), \quad t > 0, \\
u(0) = 1,
\]
\[
u(t) = 0, \quad -1 < t < 0.
\]
From (H2) it follows that there exists $K > 0$ such that $|u(t)| \leq K$, $t \geq 0$; again cf. [4, Chapter 7]. For this $K$, and $\rho^*$ and $B$ as in Remark 1, fix $\varepsilon > 0$ such that
\[
\varepsilon \int_0^\infty M(t) dt \leq \rho^* < (4BK)^{-1};
\]
note that $\rho^* \leq B$.

Let $X(B)$ denote the set of real functions $z$ continuous on $[-1, \infty)$ such that $z(t) = y^*(t)$, $-1 \leq t \leq 0$, where $y^*(t)$ is the periodic solution of (3) as described in Remark 1, and $|z(t)| \leq 2B$, $t \geq 0$. With the topology of uniform convergence on compact subsets of $[-1, \infty)$, the set $X$ of all real functions continuous on $[-1, \infty)$ is a locally convex linear topological space over the reals, and clearly $X(B) \subset X$.

Define the map $T$ on $X(B)$ to $X$ by
\[
(Tz)(t) = y^*(t) + \frac{1}{\varepsilon} \int_0^t u(t - s) N(s, \varepsilon z(s)) ds, \quad t > 0,
\]
\[
= y^*(t), \quad -1 \leq t \leq 0,
\]
for any $z \in X(B)$.

Using (5) with (H1) and the boundedness property of $u(t)$, we have
\[
(Tz)(t) \leq B + 4KB^2 \varepsilon \int_0^t M(s) ds \leq 2B, \quad t \geq 0;
\]
therefore $Tz \in X(B)$.

Using (6) and the properties of $u(t)$ given in (4) it follows that
\[
\frac{d}{dt} (Tz)(t) = y^{**}(t) + \frac{1}{\varepsilon} N(t, \varepsilon z(t)) + \frac{1}{\varepsilon} \int_0^t u'(t - s) N(s, \varepsilon z(s)) ds
\]
\[
= y^{**}(t) + \frac{1}{\varepsilon} N(t, \varepsilon z(t)) + \frac{1}{\varepsilon} \int_0^t L(u_{t-s}) N(s, \varepsilon z(s)) ds.
\]
So since $L$ is bounded, $z \in X(B)$, $u(t)$ is bounded for $t > 0$, and $N$ satisfies the conditions in $(H_1)$, it follows that there exists a constant $C(\varepsilon)$ and that

$$\frac{d}{dt}(Tz)(t) \leq C(\varepsilon), \quad t > 0.$$  

By a standard argument using the Ascoli-Arzela theorem, it then follows that $TX(B)$ is precompact in the topology of $X$, and by the Schauder-Tychonov fixed point theorem, there exists a $z^* \in X(B)$ such that

$$z^*(t) = y^*(t) + \frac{1}{\varepsilon} \int_0^t u(t-s)N(s, \varepsilon z^*(s)) \, ds, \quad t > 0,$$

$$z^*(t) = y^*(t), \quad -1 \leq t \leq 0.$$  

Since $u(t)$ is a fundamental solution for $(3)$, it follows that $z^*(t)$ solves

$$z'(t) = L(z_t) + \frac{1}{\varepsilon} N(t, \varepsilon z_t), \quad t > 0,$$

$$z(t) = y^*(t), \quad -1 \leq t \leq 0,$$

and so $y(t) = \varepsilon z^*(t)$ solves $(2)$ for $t > 0$ with $y(t) = \varepsilon y^*(t)$ for $-1 \leq t \leq 0$. If

$$R_0(t) = \frac{1}{\varepsilon} \int_0^t u(t-s)N(s, \varepsilon z^*(s)) \, ds, \quad t \geq 0,$$

then $z^*(t) = y^*(t) + R_0(t), \ t \geq 0$, and using the properties of $u$ and $N$ and the fact that $z^* \in X(B)$ it follows that

$$|R_0(t)| \leq 4B^2K \varepsilon \int_0^t M(s) \, ds \leq \frac{\rho^*}{2}, \quad t \geq 0.$$  

So

$$|z^*(t) - y^*(t)| \leq \rho^*/2, \quad t \geq 0.$$  

But using the properties of $y^*(t)$ mentioned in Remark 1, there exists $t_n \to \infty$ as $n \to \infty$, $t_{n+1} > t_n$, such that $y^*(t_n) \geq \rho^*, \ n = 1, 2, \ldots$. Using $(8)$ it follows easily that

$$z^*(t_n) \geq \rho^*/2, \quad n = 1, 2, \ldots.$$  

Similarly, there exists a sequence $\tau_n \to \infty$ as $n \to \infty$, $\tau_{n+1} > \tau_n$, such that $y^*(\tau_n) \leq -\rho^*$, and so

$$z^*(\tau_n) \leq -\rho^*/2, \quad n = 1, 2, \ldots.$$  

Thus the solution $y(t) = \varepsilon z^*(t) \equiv w(t)$ of $(2)$ is oscillatory. Now define $\varepsilon_0$ to be the supremum of the set of all $\varepsilon > 0$ for which this argument holds. Since for such $\varepsilon > 0$, $|w(t)| \leq \varepsilon B$, with $B$ as in Remark 1, and if we take $\delta_0 = 2\varepsilon_0 B$, our theorem is proved. Note that from $(5)$, $\varepsilon_0 \leq (4BKM)^{-1}$, where $M = \int_0^\infty M(t) \, dt$.

**Remark 3.** If $\beta$ is as in $(H_2)$, it can be shown that the $t_n$ and $\tau_n$ in our proof above can be chosen such that

$$t_{n+1} - t_n \leq 2\pi/\beta, \quad \text{and} \quad \tau_{n+1} - \tau_n \leq 2\pi/\beta.$$  

This follows because $y^*(t)$ can be chosen to be a linear combination of $\sin \beta t$ and $\cos \beta t$. We omit the details.
We now return to the delay logistic equation (1) with $b > 1$. If we make the change of variables $x(t) = N(t) - a/(b + 1)$ (1) becomes

$$x'(t) = -(a/(b + 1) + x(t))(bx(t) + x(t - 1)),$$

and the linear part of (9) is the equation

$$x'(t) = -(a/(b + 1))(bx(t) + x(t - 1)).$$

It is not difficult to see that all roots of the characteristic equation for (10) have negative real part, cf. [5]. From a result in [3], it also follows that if

$$a > (b + 1)/m(b),$$

where $m(b)$ is the unique root of $b = m (\log m - 1)$, then all roots of this characteristic equation are nonreal. A direct examination of this characteristic equation also shows that all nonreal roots must be simple.

Under the change of variable $y(t) = x(t) \exp(\mu t)$, where $\mu$ is a real constant, (9) becomes

$$y'(t) = A(\mu)y(t) + B(\mu)y(t - 1) + f(y(t), y(t - 1)) \exp(-\mu t)$$

where $A(\mu) = \mu - ab/(b + 1)$, $B(\mu) = -ae^\mu/(b + 1)$, and

$$f(y,z) = -(by^2 + yze^\mu).$$

It is easy to see that if $\alpha$ is the real part of a root of the characteristic equation for (9), then $\mu + \alpha$ is the real part of a corresponding root of the characteristic equation for the linear part of (12), namely

$$y'(t) = A(\mu)y(t) + B(\mu)y(t - 1).$$

So if we choose $\mu = -\max\{\Re \lambda: \lambda$ is a root of the characteristic equation for (10)$\}$, then the characteristic equation for (13) has pure imaginary roots $\pm i\beta$, $\beta > 0$, which are simple if (11) holds. Also all other roots of this equation for (13) have negative real parts. Clearly $\mu > 0$. So we see that all the hypotheses of Theorem 1 are satisfied for (11) and we have the following.

**Theorem 2.** If $b > 1$ and $a > (b + 1)/m(b)$, where $m(b)$ is as defined above, then there exist oscillatory solutions of (9) of arbitrarily small amplitude; i.e., there exist solutions of (1) which oscillate about $a/(b + 1)$.

The proof of this theorem now follows easily, since by Theorem 1, there exist such oscillatory solutions $y(t)$ of (12) and so the corresponding solutions $x(t) = y(t) \exp(-\mu t)$ are also oscillatory.

An open question presents itself: under the hypotheses of Theorem 2, are all solutions of (9) oscillatory?

**Appendix.** In the strict sense, the initial function on $[-1,0]$ for the equation defining $u(t)$ in (4) is not in $C$. What is really involved here (a point not entirely clear in [4]) is that $u(t)$ solves the initial value problem

$$u'(t) = \int_{-t}^{0} u(t + s) \, d\eta(s), \quad 0 \leq t < 1,$$

$$= \int_{-1}^{0} u(t + s) \, d\eta(s), \quad t \geq 1,$$

$$u(0) = 1,$$
where $\eta(s)$ is a function of bounded variation which by the Riesz representation theorem characterizes $L$; i.e. is such that $L(\phi) = \int_{-1}^{0} \phi(s) \, d\eta(s)$ for $\phi \in C$. This initial value problem can be shown to have a solution in a fairly standard way such as by the method of successive approximations.

REFERENCES


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