A CHARACTERIZATION OF THE GENERALIZED VERONESE SURFACES

TAKEHIRO ITOH

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ABSTRACT. We prove that compact m-regular minimal surfaces in a sphere are generalized Veronese surfaces if the Gaussian curvature satisfies an inequality.

The Gaussian curvature of a surface is an intrinsic value. It is well known that minimal surfaces of constant curvature in a sphere are rigid [1]. These surfaces are called generalized Veronese surfaces in [3 or 5]. Now, we may conjecture that generalized Veronese surfaces can be characterized only by the Gaussian curvature.

In this paper, we will prove that this conjecture is true for m-regular minimal surfaces (m-regular means that the kth normal space N_k satisfies dim N_k > dim N_{k-1} for k = 1, 2, ..., m, where dim N_0 = 0, and 0-regular mapping means an ordinary regular one). That is, we prove the following:

THEOREM. Let M be a compact connected oriented surface minimally immersed in a unit sphere through the m-regular immersion. If its Gaussian curvature K satisfies

\[
\frac{2}{(m+2)(m+3)} \leq K \leq \frac{2}{(m+1)(m+2)}, \quad 0 \leq m,
\]

then M is a generalized Veronese surface.

1. Preliminaries. Let \( \tilde{M} \) be a \((2+\nu)\)-dimensional Riemannian manifold of constant curvature \( \tilde{c} \), and \( M \) a 2-dimensional Riemannian manifold which is immersed in \( \tilde{M} \) by the immersion \( x : M \rightarrow \tilde{M} \). Let \( (e_1, e_2, \ldots, e_{2+\nu}) \) be an orthonormal frame field over \( \tilde{M} \) such that \( (e_1, e_2) \) is an orthonormal frame field over \( M \). Let \( \omega_A \) and \( \omega_{AB} \) be basic and connection forms with respect to the above orthonormal frames. Then, as is well known, we have

\[
\omega_A = 0, \quad \omega_{i\alpha} = \sum_j h^0_{ij} \omega_j, \quad h^0_{ij} = h^0_{ji},
\]

\[
d\omega_i = \omega_{ij} \wedge \omega_j, \quad i \neq j,
\]

\[
d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \Omega_{ij}, \quad \Omega_{ij} = \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,
\]

\[
R_{ijkl} = \tilde{c}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^{\alpha}_{ik} h^0_{jl} - h^{\alpha}_{il} h^0_{jk}),
\]
\[ d\omega_{\alpha\beta} = \sum_{\gamma} \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \Omega_{\alpha\beta}, \quad \Omega_{\alpha\beta} = \frac{1}{2} \sum_{i,j} R_{\alpha\beta ij}\omega_i \wedge \omega_j, \]

\[ R_{\alpha\beta ij} = \sum_k (h_{ik}^\alpha h_{kj}^\beta - h_{jk}^\alpha h_{ki}^\beta), \]

where we use the following convention on the ranges of indices:

\[ 1 \leq A, B, \ldots \leq 2 + \nu, \quad 1 \leq i, j, \ldots \leq 2, \quad 3 \leq \alpha, \beta, \ldots \leq 2 + \nu. \]

\( M \) is said to be minimal if its mean curvature \( \frac{1}{2} \sum_i h_{ii}^\alpha \epsilon_\alpha \) vanishes identically, i.e., if \( \text{trace } H_\alpha = 0 \) for all \( \alpha, H_\alpha = (h_{ii}^\alpha) \).

Let \( M_p \) be the tangent space at \( p \) to \( M \) and \( N(p) \) the normal space of \( M \) at \( p \).

We can write the (first) shape operator (the second fundamental form) \( \varphi_1 \) as

\[ \varphi_1(X, Y) = \sum h_{ij}^\alpha \omega_i(X) \omega_j(Y) e_\alpha, \quad X, Y \in M_p. \]

Now we define \( h_{i_1 \ldots i_k}^\alpha, \) \( 2 \leq k, \) by

\[ \sum h_{i_1 \ldots i_k \omega_m}^\alpha := \sum h_{i_1 \ldots i_k}^\alpha \omega_m + \sum h_{i_1 \ldots i_{j-1}, m_{j+1} \ldots i_k}^\alpha \omega_{m_j} + \sum h_{i_1 \ldots i_k}^\beta \omega_\beta. \]

Then we can define the \( k \)th shape operator (the \( k \)th second fundamental form) \( \varphi_k \) as the multilinear mapping from \( M_p \times \cdots \times M_p \) into \( N(p) \) by

\[ \varphi_k(X_1, X_2, \ldots, X_{k+1}) := \sum h_{i_1 \ldots i_{k+1} \omega_i}^\alpha(X_1) \ldots \omega_{i_{k+1}}(X_{k+1}) e_\alpha, \]

where \( X_j \in M_p, j = 1, 2, \ldots, k + 1. \) We define the \( k \)th normal space \( N_k(p) \) of \( M \) at \( p \) as

\[ N_k(p) := \text{Span}\{\varphi_1(X_1, X_2), \varphi_2(X_1, X_2, X_3), \ldots, \varphi_k(X_1, \ldots, X_{k+1})\}, \]

where \( X_j \in M_p, j = 1, 2, \ldots, k + 1. \) The immersion is said to be \( m \)-regular if \( \dim N_k(p) > \dim N_{k-1}(p) \) at each point \( p \in M \) for all \( k = 1, 2, \ldots, m, \) where \( \dim N_0(p) = 0. \) A 0-regular mapping means an ordinary one, so an immersion is 0-regular.

2. Compact minimal positive curvature surfaces in a space form. Let \( M \) be a compact connected oriented surface which is minimally immersed in a \((2 + \nu)\)-dimensional Riemannian manifold \( M \) by the \( m \)-regular immersion. We suppose that the Gaussian curvature \( K \) of \( M \) is positive. Let \( U \) be a neighborhood of a point \( p \in M \) in which there exist isothermal coordinates \((u, v)\) and a frame field \((e_1, e_2)\) such that

\[ (2.1) \quad ds^2 = E(du^2 + dv^2), \quad \omega_1 = \sqrt{E} du, \quad \omega_2 = \sqrt{E} dv, \]

where \( ds \) is the line element of \( M \) and \( E = E(u, v) \) is a positive function on \( U. \)

Since \( M \) is minimal in \( U, \) we may write

\[ \omega_1 = f_\alpha \omega_1 + g_\alpha \omega_2, \quad \omega_2 = g_\alpha \omega_1 - f_\alpha \omega_2, \quad 2 < \alpha, \]

where \( f_\alpha \) and \( g_\alpha \) are functions on \( U. \) Then, using the structure equations, we easily see that the complex valued function

\[ (2.2) \quad w_1(z, \bar{z}) = E^2(|G_1|^2 - |F_1|^2) + 2iE^2(G_1, F_1) \]

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is holomorphic in $z = u + iv$, where $F_1 = \sum f_\alpha e_\alpha$ and $G_1 = \sum g_\alpha e_\alpha$. For a tangent vector $X = \cos \theta \cdot e_1 + \sin \theta \cdot e_2$ of $M_p$, we have the mapping $\varphi_1$ from $M_p$ into $N(p)$ as follows:

\begin{equation}
(2.3) \quad \varphi_1(X) := \varphi_1(X, X) = \cos 2\theta \cdot F_1 + \sin 2\theta \cdot G_1,
\end{equation}

where $F_1 = \sum f_\alpha e_\alpha = \sum h_{11}^2 e_\alpha$ and $G_1 = \sum g_\alpha e_\alpha = \sum h_{12}^2 e_\alpha$. Then we have

**Lemma 1.** At each point $p$ of $M$, the image of the tangent unit circle $S^1_p$ under the mapping $\varphi_1$ is a point or a circle according as $\dim N_1(p) = 0$ or $\neq 0$, where $S^1_p = \{X \in M_p \mid |X| = 1\}$.

**Proof.** We easily see that $|w_1(z, \bar{z})|^2/E^4$ is a differentiable function on $M$. Since $M$ is compact, $|w_1(z, \bar{z})|^2/E^4$ takes the maximum $A$ at some point $p_0$ of $M$. If $A > 0$, then there exists a neighborhood $U$ of $p_0$ in which $|w_1(z, \bar{z})|^2/E^4 > 0$, and there exist isothermal coordinates $(u, v)$ and a frame field satisfying (2.1). Then from (2.2) we have

\begin{equation}
(2.4) \quad \Delta \log(|w_1(z, \bar{z})|^2/E^4) = -4\Delta \log E = 8EK, \quad \Delta = \partial^2/\partial u^2 + \partial^2/\partial v^2,
\end{equation}

because $K$ is given by $K = -(1/2E)\Delta \log E$. It follows from $K > 0$ and (2.4) that the function $\tilde{w}_1 = \log(|w_1(z, \bar{z})|^2/E^4)$ is a subharmonic function on $M$. Since $\tilde{w}_1$ takes the maximum $\log A$ at $p_0 \in U$, it must be constant $\log A$ on $U$. Furthermore, since $M$ is connected, $\tilde{w}_1$ is constant on $M$. It follows from this fact and (2.4) that $K$ is identically zero on $M$, which contradicts $K > 0$. Thus the function $w_1(z, \bar{z}) = 0$, so $F_1$ and $G_1$ are orthonormal vectors in $N_1(p)$. Hence, we have proved our assertion.

If $m = 1$, that is, $\dim N_1(p) \neq 0$ at each point $p \in M$, then, by Lemma 1, we choose a neighborhood $U$ of $p$ in which there exist isothermal coordinates $(u, v)$ and frame fields satisfying (2.1) and

\begin{equation}
(2.5) \quad \omega_{13} = k_1 \omega_1 = \omega_{24}, \quad \omega_{1\beta} = \omega_{2\beta} = 0,
\omega_{23} = -k_1 \omega_2 = -\omega_{14}, \quad 4 < \beta,
\end{equation}

where $k_1$ is a positive differentiable function on $M$. Using the structure equations, from (2.5) we have

\begin{equation}
(2.6) \quad \omega_{34} = 2\omega_{12} - (\log k_1)\omega_1 + (\log k_1)\omega_2,
\end{equation}

where $d(\log k_1) = \sum_j (\log k_1)\omega_j$. Furthermore, by (2.5) we may put

\begin{equation}
(2.7) \quad k_1 \omega_{3\beta} = f_\beta \omega_1 + g_\beta \omega_2, \quad k_1 \omega_{4\beta} = g_\beta \omega_1 - f_\beta \omega_2, \quad 4 < \beta,
\end{equation}

and define two normal vectors $F_2 = \sum f_\beta e_\beta$ and $G_2 = \sum g_\beta e_\beta$ on $U$. Then, from $\varphi_2$ we have the mapping $\tilde{\varphi}_2$ from $M_p$ into $N_2(p)$ as

\begin{equation}
(2.8) \quad \tilde{\varphi}_2(X) := \varphi_2(X, X, X) = \cos 3\theta \cdot F_2 + \sin 3\theta \cdot G_2,
\end{equation}

for $X = \cos \theta \cdot e_1 + \sin \theta \cdot e_2 \in M_p, p \in U$. Using the structure equations, by (2.5) and (2.7) the complex valued function

\begin{equation}
\omega(z, \bar{z}) = E^3(|G_2|^2 - |F_2|^2) + 2iE^3\langle F_2, G_2 \rangle
\end{equation}

is holomorphic in $z$. Furthermore, $|\omega(z, \bar{z})|^2/E^6$ is a differentiable function on $M$. Hence, in the same way as Lemma 1, we can prove the following
Lemma 2. If \( m = 1 \), then, at each point \( p \in M \), the image of a unit tangent circle \( S^1_p \) under \( \varphi_2 \) is a point or a circle according as \( \dim N_2(p) - \dim N_1(p) = 0 \) or \( \neq 0 \).

If \( m = 2 \), that is, \( \dim N_2(p) - \dim N_1(p) > 0 \) at each point \( p \in M \), then, by Lemma 1 and Lemma 2, we choose a neighborhood \( U \) of \( p \) in which there exist isothermal coordinates \( (u,v) \) and frame fields satisfying (2.1), (2.5) and

\[
\begin{align*}
k_1\omega_{35} &= k_2\omega_1 = k_1\omega_{46}, & \omega_3\gamma &= \omega_4\gamma = 0, \\
k_1\omega_{36} &= k_2\omega_2 = -k_1\omega_{45}, & 6 < \gamma,
\end{align*}
\]

where \( k_2 \) is a positive differentiable function on \( M \). Let \( d(\log k_2) = \sum (\log k_2) j \omega_j \); then from (2.9) we have

\[
\omega_{56} = 3\omega_{12} - (\log k_2) \omega_1 + (\log k_2) \omega_2
\]

and we may write

\[
k_2\omega_{5\gamma} = f_7\omega_1 + g_\gamma\omega_2, \quad k_2\omega_{6\gamma} = g_\gamma\omega_1 - f_7\omega_2, \quad 6 < \gamma.
\]

From the third shape operator \( \varphi_3 \) we have the mapping \( \tilde{\varphi}_3 \) from \( M_p \) into \( N_3(p) \) as

\[
\tilde{\varphi}_3(X) := \varphi_3(X,X,X) = \cos 4\theta \cdot F_3 + \sin 4\theta \cdot G_3,
\]

for \( X = \cos \theta \cdot e_1 + \sin \theta \cdot e_2 \in M_p, p \in M \), where \( F_3 = \sum f_7 e_7 \) and \( G_3 = \sum g_\gamma e_\gamma \) are normal vector fields on \( U \). Furthermore, from (2.9), (2.10) and (2.11) we see that the complex function

\[
w_3(z, \overline{z}) = E^4(|G_3|^2 - |F_3|^2) + 2iE^4(F_3, G_3)
\]

is holomorphic in \( z \).

Continuing this way, we have the following:

Theorem 1. Let \( M \) be a connected compact oriented surface of positive curvature minimally immersed in a \( (2 + \nu) \)-dimensional space form \( \tilde{M} \) of constant curvature \( c \). If the immersion is \( m \)-regular, then, at each point on \( M \), the image of a unit tangent circle \( S^1_p \) under the mapping \( \tilde{\varphi}_n \) (the \( n \)th shape operator) is a circle for any \( n = 1, 2, \ldots, m \), and the image of \( S^1_p \) under \( \tilde{\varphi}_{m+1} \) is a point or a circle according as \( \dim N_{m+1}(p) - \dim N_m(p) = 0 \) or \( \neq 0 \).

Proof. For \( n = 1, 2 \), we have proved our assertion in Lemma 1 and Lemma 2. By induction on \( n \), we will prove our assertion. So we assume that the above assertion holds for all \( t \leq n-1 < m, 3 \leq n \). Then we can choose a neighborhood \( U \) of a point \( p \in M \) in which there exist isothermal coordinates \( (u,v) \) and frame fields satisfying (2.1) and

\[
(2.12)_{n-1} \quad \begin{cases}
k_{t-1} \omega_{\alpha_1 \beta_1} = k_t \omega_1 = k_{t-1} \omega_{\alpha_2 \beta_2}, & \omega_{\alpha_1 \gamma} = \omega_{\alpha_2 \gamma} = 0, \\
k_{t-1} \omega_{\alpha_1 \beta_2} = k_t \omega_2 = -k_{t-1} \omega_{\alpha_2 \beta_1}, & 2t + 2 < \gamma, \\
\alpha_1 = 2t - 1, & \alpha_2 = 2t, \\
\beta_1 = 2t + 1, & \beta_2 = 2t + 2,
\end{cases}
\]

where \( k_0 = 1 \) and \( k_t \) (\( 1 \leq t \leq n-1 \)) are positive differentiable functions on \( M \). Using the structure equations, from (2.12)\(_{n-1}\) we have

\[
(2.13)_{n-1} \quad \omega_{\beta_1 \beta_2} = (t + 1)\omega_{12} - (\log k_t) \omega_1 + (\log k_t) \omega_2,
\]
where $\beta_1 = 2t + 1$, $\beta_2 = 2t + 2$, and $d(\log k_t) = \sum (\log k_t)_j \omega_j$ for $t = 1, 2, \ldots, n - 1$. Furthermore, from $(2.12)_{n-1}$ we may write

$$
k_{n-1} \omega_{a_1} \gamma = f_\gamma \omega_1 + g_\gamma \omega_2, \quad a_1 = 2n - 1, \\
k_{n-1} \omega_{a_2} \gamma = g_\gamma \omega_1 - f_\gamma \omega_2, \quad a_2 = 2n, \quad 2n < \gamma.
$$

From the $n$th shape operator $\varphi_n$ we have the mapping $\tilde{\varphi}_n$ from $M_p$ into $N_n(p)$ as $\tilde{\varphi}_n(X) := \varphi_n(X, \ldots, X)$ for $X \in M_p$. For a unit tangent vector $X = \cos \theta \cdot e_1 + \sin \theta \cdot e_2$ at $p$ to $M$, we have

$$
\tilde{\varphi}_n(X) = \cos(n + 1) \theta \cdot F_n + \sin(n + 1) \theta \cdot G_n,
$$

where $F_n = \sum f_\gamma e_\gamma$ and $G_n = \sum g_\gamma e_\gamma$ are normal vector fields on $U$. Using the structure equations, from $(2.12)_{n-1}$ and $(2.13)_{n-1}$ we see that the complex valued function

$$
w_n(z, \bar{z}) = \frac{2}{(2n+2)} |G_n|^2 - |F_n|^2 + 4 \langle G_n, F_n \rangle
$$

is holomorphic in $z = u + iv$. As stated above, we easily see that

$$
\frac{|w_n(z, \bar{z})|^2}{E^{2n+2}} = |G_n|^2 - |F_n|^2 + 4 \langle G_n, F_n \rangle
$$

is a differentiable function on $M$. Hence, in the same way as in the proof of Lemma 1, at each point $p \in M$, the image of $S^n_p$ under $\tilde{\varphi}_n$ is a circle, because $n \leq m$. Furthermore, when $n = m + 1$, by the above consideration we see that the image of $S^n_p$ under $\tilde{\varphi}_{m+1}$ is a point or a circle according as $\dim N_{m+1}(p) - \dim N_m(p) = 0$ or $\neq 0$. Thus we have proved our assertion.

Now we can prove the following:

**THEOREM 2.** Let $M$ be a connected compact oriented surface minimally im-

mersed in a $(2 + \nu)$-dimensional space form $M$ of constant curvature $c$. If the

immersion is m-regular and the Gaussian curvature $K$ satisfies the inequality

$$
\frac{2c}{(m+2)(m+3)} \leq K \leq \frac{2c}{(m+1)(m+2)},
$$

then $M$ is of constant curvature $2c/((m+2)(m+3))$ or $2c/((m+1)(m+2))$.

**PROOF.** First we consider the case that $\dim N_{m+1}(p) = \dim N_m(p)$ at each

point $p \in M$, so $k_{m+1}^2 = 0$ on $M$ by Theorem 1. Using the structure equations,

from $(2.5)$, $(2.12)$ and $(2.13)$ we have

$$
\Delta \log k_t = E\{(t+1)K - 2k_t^2/k_{t-1}^2 + 2k_{t+1}^2/k_t^2\}, \quad t = 1, 2, \ldots, m - 1, \\
\Delta \log k_m = E\{(m+1)K - 2k_m^2/k_{m-1}^2\}, \quad \Delta = \partial^2/\partial u^2 + \partial^2/\partial v^2,
$$

which imply

$$
\Delta \log(k_1 \cdot k_2 \cdots k_m) = E \left\{ \frac{m(m+3)}{2} K - 2k_1^2 \right\} = E \left\{ \frac{(m+1)(m+2)}{2} K - c \right\},
$$

because $K = c - 2k_1^2$. Since $K \leq 2c/((m+1)(m+2))$, $(2.16)$ implies that the

function $\log(k_1 \cdot \cdots \cdot k_m)$ is a superharmonic function on $M$. Since $M$ is compact,

$\log(k_1 \cdot \cdots \cdot k_m)$ must be constant on $M$, which, together with $(2.16)$, implies $K = 2c/((m+1)(m+2))$ identically on $M$. 

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Next, we consider the case that there exists a point $p_0 \in M$ such that $\dim N_{m+1}(p_0) > \dim N_m(p_0)$. We choose a neighborhood $U$ of $p_0$ in which there exist isothermal coordinates $(u, v)$ and frame fields satisfying (2.1) and (2.12)$_{m+1}$. We must remark that $k^2_{m+1}$ is a differentiable function on $M$ and vanishes at a point where $\dim N_{m+1}(p) = \dim N_m(p)$. On $U$ we can consider the image of $S^1_p$ under the mapping $\tilde{\varphi}_{m+2}$. For the same reason as above, it is a point or a circle according as $\dim N_{m+2}(p) - \dim N_{m+1}(p) = 0$ or $\neq 0$. Using the structure equations, from (2.5), (2.12)$_{m+1}$ and (2.13)$_{m+1}$ we have

\begin{equation}
(2.17) \quad \Delta \log k_t = E\{(t + 1)K - 2k^2_t/k^2_{t-1} + 2k^2_{t+1}/k^2_t\}, \quad t = 1, \ldots, m+1,
\end{equation}

where $k^2_{m+2} = \sum_{2(m+2)<\gamma} f^2_\gamma$ is the square of the radius of the image of $S^1_p$. From (2.16) we have

\begin{equation}
(2.18) \quad \Delta(\log k_1 \cdots k_{m+1}) = E\left\{\frac{(m+1)(m+4)}{2}K - 2k^2_1 + 2k^2_{m+2}/k^2_{m+1}\right\}
\end{equation}

which, together with $K \geq 2c/((m+2)(m+3))$, implies $\log(k_1 \cdots k_{m+1})$ is a subharmonic function on $U$. Here we may assume that the differentiable function $k^2_1 \cdots k^2_{m+1}$ takes the maximum value at $p_0$. Then $\log(k_1 \cdots k_{m+1})$ takes its maximum at $p_0$ in $U$, so it must be constant on $U$. Hence (2.18) implies that $K = 2c/((m+2)(m+3))$ on $U$. Since $M$ is connected, $K = 2c/((m+2)(m+3))$ on $M$. Thus we have proved our assertion.

We see that our main theorem is obtained as a corollary of the results in [3] and the above Theorem 2.

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Institute of Mathematics, University of Tsukuba, Tsukuba-shi, Ibaraki, 305 Japan