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THE VISIBILITY AXIOM ON A HADAMARD MANIFOLD WHOSE GEODESIC FLOW IS OF ANOSOV TYPE

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ABSTRACT. For a Hadamard manifold M, the set of points at infinity $M(\infty)$ is defined. If the geodesic flow on the unit tangent bundle of M is of Anosov type, then with a certain curvature condition M satisfies the Visibility Axiom. To prove this result, we use the Tits metric on $M(\infty)$.

1. Introduction. A connected, simply connected complete Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold. Throughout this paper, let M denote a Hadamard manifold of dimension $n \geq 2$. Let $\langle\ ,\ \rangle$ denote the inner product on M, and $d(\ ,\ )$ be the distance function on M. Let TM and SM denote respectively the tangent bundle and the unit tangent bundle of M, and π be the natural projection map onto M in either case. For any vector $v \in TM$ let γ_v be the unique maximal geodesic of M such that $\gamma_v'(0) = v$, where $\gamma_v'(t)$ is the velocity of γ_v at time t.

DEFINITION 1.1. For any $t \in R$ we define a map $T_t\colon TM \to TM$ as follows: Given a vector $v \in TM$, let $T_t v := \gamma_v'(t)$. The collection of maps T_t is called the *geodesic flow* on TM.

Since the geodesic flow leaves SM invariant, its restriction to SM is called the geodesic flow on SM. If V denotes the vector field on TM defined by the geodesic flow, then the restriction of V to SM is a tangent vector field on SM.

DEFINITION 1.2. Let T_t be the geodesic flow on SM. The flow is said to be *of Anosov type*, if the following conditions are satisfied:

For each $v \in SM$ the tangent space $(SM)_v$ splits into a direct sum as follows.

$$(SM)_v = X_s^*(v) \oplus X_u^*(v) \oplus Z(v)$$

($\dim X_s^*(v) > 0$, $\dim X_u^*(v) > 0$, $\dim Z(v) = 1$), where $Z(v)$ is generated by $V(v)$, and there exist positive numbers a, b, c, such that

(1) for any $\xi \in X_s^*(v)$ $\|dT_t\xi\| \leq a\|\xi\|e^{-ct}$ for $t \geq 0$, $\|dT_t\xi\| \geq b\|\xi\|e^{-ct}$ for $t \leq 0$,

(2) for any $\eta \in X_u^*(v)$ $\|dT_t\eta\| \leq a\|\eta\|e^{ct}$ for $t \leq 0$, $\|dT_t\eta\| \geq b\|\eta\|e^{ct}$ for $t \geq 0$, where $\|\ \|$ is the natural norm on $(SM)_v$ defined in §2.

DEFINITION 1.3. Let M be a Hadamard manifold. Two unit speed geodesics c_1, c_2 are called *asymptotic*, if there is a constant $b_1 > 0$, such that $d(c_1(t), c_2(t)) \leq b_1$ for all $t \geq 0$. The equivalence classes of this relation are called *points at infinity*

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and are denoted by $M(\infty)$. For a unit speed geodesic $c\colon R \to M$, let $c(\infty) \in M(\infty)$ be the corresponding class, and $c(-\infty) \in M(\infty)$ the class of the reversed geodesic $t \mapsto c(-t)$.

The main purpose of this paper is to prove the following theorem.

THEOREM 4.3. *Let M be a Hadamard manifold whose sectional curvature K satisfies $-k^2 < K \le 0$ for some constant $k > 0$, and assume that the geodesic flow on SM is of Anosov type. Then for any different points $z, w \in M(\infty)$, there exists a unit speed geodesic c which satisfies $c(-\infty) = z$, $c(\infty) = w$.*

Let p be a point of M distinct from points $x, y \in M$. The *angle* subtended by x, y at p is $\measuredangle_p(x, y) := \measuredangle(\gamma'_{px}(0), \gamma'_{py}(0))$, where γ_{px}, γ_{py} are geodesics from p to x, y respectively.

DEFINITION 1.4. Let M be a Hadamard manifold. M satisfies the *Visibility Axiom* if given $p \in M$ and $\varepsilon > 0$ there exists a number $r = r(p, \varepsilon)$ with the property:

If $\gamma\colon [a, b] \to M$ is a geodesic segment such that $d(p, \gamma([a, b])) \ge r$, then

$$\measuredangle_p(\gamma(a), \gamma(b)) \le \varepsilon.$$

From Theorem 4.3, we obtain the following.

COROLLARY 4.4. *Let M satisfy the hypotheses of Theorem* 4.3. *Then M satisfies the Visibility Axiom.*

REMARK 1.5. If the sectional curvature K of a Hadamard manifold M satisfies $K \le c < 0$ for some constant c, then M satisfies the Visibility Axiom [**7**, p. 61].

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2. Anosov geodesic flows. For a manifold X and $p \in X$, let X_p denote the tangent space at p, and TX denote the tangent bundle of X. For each $v \in TM$, $d\pi\colon (TM)_v \to M_{\pi(v)}$ is linear. We define a *connection map* $K\colon T(TM) \to TM$ such that for each $v \in TM$, $K\colon (TM)_v \to M_{\pi(v)}$ is linear in the following way. Given a vector $\xi \in (TM)_v$, let $Z\colon (-\varepsilon, \varepsilon) \to TM$ be a C^∞ curve with $Z(0) = v$ and $Z'(0) = \xi$. Let $\alpha = \pi \circ Z\colon (-\varepsilon, \varepsilon) \to M$. We define $K(\xi) := DZ/dt(0) \in M_{\pi(v)}$, where $DZ/dt(0)$ is the covariant derivative of Z along α evaluated at $t = 0$. $K(\xi)$ does not depend on the curve Z chosen.

One may define a natural *inner product* on TM by the following. Given vectors $\xi, \eta \in (TM)_v$ let $\langle \xi, \eta \rangle_v := \langle d\pi\xi, d\pi\eta \rangle_{\pi(v)} + \langle K\xi, K\eta \rangle_{\pi(v)}$. For any $v \in SM$ and any $\xi \in (SM)_v$, let $||\xi|| := \langle \xi, \xi \rangle^{1/2}$.

For any number $t \ne 0$, we define a linear map $\xi \mapsto \xi_t\colon (TM)_v \to (TM)_v$ for every $v \in TM$ as follows. Given a vector $v \in TM$ and a vector $\xi \in (TM)_v$, let $\xi_t \in (TM)_v$ be the unique vector such that $d\pi(\xi_t) = d\pi(\xi)$ and $d\pi \circ dT_t(\xi_t) = 0$.

DEFINITION 2.1. For every $v \in SM$ let $X_s(v) := \{\xi \in (SM)_v | \langle \xi, V(v) \rangle = 0$ and $\xi_t \to \xi$ as $t \to +\infty\}$. Let $X_u(v) := \{\xi \in (SM)_v | \langle \xi, V(v) \rangle = 0$ and $\xi_t \to \xi$ as $t \to -\infty\}$.

REMARK 2.2. For nonpositively curved manifolds one can define $X_s(v)$ and $X_u(v)$ equivalently as follows:

$$X_s(v) = \{\xi \in (SM)_v|\ ||d\pi \circ dT_t(\xi)|| \le c \text{ for some } c > 0 \text{ and all } t \ge 0\},$$

$$X_u(v) = \{\xi \in (SM)_v|\ ||d\pi \circ dT_t(\xi)|| \le c \text{ for some } c > 0 \text{ and all } t \le 0\}.$$

PROPOSITION 2.3 [**6**, p. 446]. *For every $v \in SM$, both $X_s(v)$ and $X_u(v)$ are $(n-1)$-dimensional vector subspaces of $(SM)_v$.*

For its proof, see the reference above.

If the geodesic flow on SM is of Anosov type, then for each $v \in SM$ the tangent space $(SM)_v$ splits as Definition 1.2.

PROPOSITION 2.4 [**6**, p. 455]. *Let M be a Hadamard manifold whose sectional curvature K satisfies $-k^2 < K \leq 0$ for some constant $k > 0$. Assume that the geodesic flow on SM is of Anosov type. Then $X_s(v) = X_s^*(v)$, and $X_u(v) = X_u^*(v)$ for every $v \in SM$.*

For its proof, see the reference above.

DEFINITION 2.5. Let $c\colon R \to M$ be a unit speed geodesic. For a finite point $x \in M$ the function

$$h_{c(\infty)}(x) := \lim_{t\to\infty}\{d(x, c(t)) - t\}$$

is well defined and is called the *Busemann function* at $c(\infty)$. For every $t \in R$ we define $H(c(t), c(\infty)) := h^{-1}_{c(\infty)}(-t)$, which is called the *horosphere* through $c(t)$ with center $c(\infty)$. Since a Busemann function is of class C^2 [**9**, p. 484], a horosphere is a C^2-submanifold of M.

DEFINITION 2.6. For $v \in SM$, take the geodesic $c\colon R \to M$ such that $c'(0) = v$. Let

$$\mathscr{H}^+(v) := \{(p, -\operatorname{grad} h_{c(\infty)}(p)) \in SM \mid p \in H(c(0), c(\infty))\}$$

and

$$\mathscr{H}^-(v) := \{(p, \operatorname{grad} h_{c(-\infty)}(p)) \in SM \mid p \in H(c(0), c(-\infty))\},$$

where $\operatorname{grad} h_{c(\infty)}$ is the gradient vector field of $h_{c(\infty)}$. Then $\mathscr{H}^+(v)$ and $\mathscr{H}^-(v)$ are C^2-submanifolds of SM with dimension $n-1$.

LEMMA 2.7. *Let $v \in SM$. Then $X_s(w) = (\mathscr{H}^+(v))_w$ for $w \in \mathscr{H}^+(v)$, and $X_u(w) = (\mathscr{H}^-(v))_w$ for $w \in \mathscr{H}^-(v)$.*

PROOF. Since $\mathscr{H}^+(w) = \mathscr{H}^+(v)$ for any $w \in \mathscr{H}^+(v)$, we prove this lemma only when $w = v$. Let $c\colon R \to M$ be the unique geodesic with $c'(0) = v$. Take $u \in M_{\pi(v)}$, such that $u \perp v$. Let $a\colon (-\varepsilon, \varepsilon) \to H(c(0), c(\infty))$ be a curve with $a'(0) = u$, and for $n \to \infty$ $a_n\colon (-\varepsilon, \varepsilon) \to S_n(c(n))$ be curves with $a_n'(0) = u$, where $S_n(c(n)) := \{p \in M \mid d(p, c(n)) = n\}$. For $s \in (-\varepsilon, \varepsilon)$, let γ_s and $\gamma_{n,s}$ be the unique unit speed geodesics from $a(s)$ to $c(\infty)$ and from $a_n(s)$ to $c(n)$ respectively. We define variations b, $b_n\colon (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \to M$ by $b(s,t) := \gamma_s(t)$ and $b_n(s,t) := \gamma_{n,s}(t)$ respectively. Let $Y := -\operatorname{grad} h_{c(\infty)}$ and $Y_n := -\operatorname{grad} d_{c(n)}$, where $d_{c(n)}\colon p \mapsto d(c(n), p)$. Then for each $(s,t) \in (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)$, we have $Y(b(s,t)) = \gamma_s'(t)$, and $Y_n(b_n(s,t)) = \gamma_{n,s}'(t)$. Then it is known that $\nabla_u Y_n(c(0)) \to \nabla_u Y(c(0))$ as $n \to \infty$ [**9**, p. 484], where $\nabla_u Y$ is the covariant derivative of Y in the direction u. Let $\xi := (u, \nabla_u Y(c(0))) \in (SM)_v$, where $d\pi\xi = u$ and $K\xi = \nabla_u Y(c(0))$. Then

$$\xi = \left(\frac{\partial b}{\partial s}(0,0),\ \frac{D}{\partial s}\frac{\partial b}{\partial t}(0,0)\right)$$

and

$$\xi_n = (d\pi(\xi_n), K(\xi_n)) = \left(u, \frac{D}{\partial s}\frac{\partial b_n}{\partial t}(0,0)\right) = (u, \nabla_u Y_n(c(0)))$$

[**6**, p. 441]. So $\xi \in (\mathscr{H}^+(v))_v$ and $\xi_n \to \xi$ as $n \to \infty$. It proves that $(\mathscr{H}^+(v))_v \subset X_s(v)$. Since both of these spaces have the same dimension $n-1$ (Proposition 2.3), we see that $(\mathscr{H}^+(v))_v = X_s(v)$. The second part of this lemma is proved similarly.

LEMMA 2.8. *Let M be a Hadamard manifold whose sectional curvature K satisfies $-k^2 < K \leq 0$ for some constant $k > 0$. Assume that the geodesic flow on SM is of Anosov type, and let $c_1, c_2 \colon R \to M$ be unit speed geodesics such that $c_1(-\infty) = c_2(-\infty)$ and $c_1(t)$, $c_2(t)$ lie on the same horosphere with center $c_1(-\infty)$ for any $t \in R$. Then for every $t \geq 0$,*

$$d^H(c_1(t), c_2(t)) \geq b(1+k^2)^{-1/2} e^{ct} d^H(c_1(0), c_2(0))$$

where d^H denotes the distance function in the corresponding horosphere.

PROOF. Since horospheres are complete C^2 submanifolds of M, for any $t \geq 0$ there exists a geodesic $\gamma_t \colon [0,1] \to H(c_1(t), c_1(-\infty))$ in $H(c_1(t), c_1(-\infty))$ such that $\gamma_t(0) = c_1(t)$, $\gamma_t(1) = c_2(t)$ and $L(\gamma_t) = d^H(c_1(t), c_2(t))$, where $L(\ )$ denotes the length of a curve. For a fixed number $t \geq 0$, we define a curve $\mu \colon [0,1] \to H(c_1(0), c_1(-\infty))$ by $\mu(s) := \pi \circ T_t(-\operatorname{grad} h_{c_1(-\infty)}(\gamma_t(s)))$. Consider a curve $W \colon [0,1] \to SM$ defined by $W(s) := (\mu(s), -\operatorname{grad} h_{c_1(-\infty)}(\mu(s)))$, and let $\xi(s) := W'(s) \in (SM)_{W(s)}$. Then from Lemma 2.7 and Proposition 2.4, we have $\xi(s) \in X_s(W(s)) = X_s^*(W(s))$ for any $s \in [0,1]$. Since the sectional curvature K satisfies $-k^2 < K \leq 0$, for any $v \in SM$ and any $\xi \in X_s(v)$ or $X_u(v)$ we have $||K\xi|| \leq k||d\pi\xi||$ [**6**, p. 448]. Now the Anosov condition implies

$$||dT_{-t}\xi(s)|| \geq b||\xi(s)||e^{ct} \quad \text{for } t \geq 0.$$

Since $dT_{-t}\xi(s) \in X_s(-\operatorname{grad} h_{c_1(-\infty)}(\gamma_t(s)))$, it follows that

$$\begin{aligned}(1+k^2)^{1/2}||d\pi \circ dT_{-t}\xi(s)|| &\geq (||d\pi \circ dT_{-t}\xi(s)||^2 + ||K \circ dT_{-t}\xi(s)||^2)^{1/2} \\ &= ||dT_{-t}\xi(s)|| \geq b||\xi(s)||e^{ct} \geq b||d\pi\xi(s)||e^{ct}.\end{aligned}$$

For $s \in [0,1]$, $\gamma_t'(s) = d\pi \circ dT_{-t}\xi(s)$ and $\mu'(s) = d\pi\xi(s)$. So we have,

$$\begin{aligned}L(\gamma_t) &= \int_0^1 ||\gamma_t'(s)||\,ds \geq b(1+k^2)^{-1/2}e^{ct}\int_0^1 ||\mu'(s)||\,ds \\ &= b(1+k^2)^{-1/2}e^{ct}L(\mu).\end{aligned}$$

Now this lemma follows from the fact $d^H(c_1(t), c_2(t)) = L(\gamma_t)$ and $d^H(c_1(0), c_2(0)) = L(\gamma_0) \leq L(\mu)$.

DEFINITION 2.9. Let γ be a geodesic of M. A vector field Y along γ is a *Jacobi field* if $(D/dt)(DY/dt) + R(\gamma', Y)\gamma' = 0$, where γ' is the velocity vector field of γ, and R is the curvature tensor of M.

PROPOSITION 2.10 [**6**, p. 455]. *Let M be a Hadamard manifold whose sectional curvature K satisfies $-k^2 < K \leq 0$ for some constant $k > 0$, and assume that the geodesic flow on SM is of Anosov type. Then there exists no nonzero perpendicular Jacobi field Y along a unit speed geodesic γ of M such that $||Y(t)||$ is bounded above for all $t \in R$.*

For its proof, see the reference above.

3. The Tits metric. In this section, every lemma is related with the Tits metric. Here its proof is omitted (see [**3**, §4]).

DEFINITION 3.1. Let M be a Hadamard manifold. Fix a point $x \in M$, and for any $z, w \in M(\infty)$ let c_1, c_2 be the unique unit speed geodesics from x to z, from x to w respectively. Then $t \mapsto (1/t)\, d^S(c_1(t), c_2(t))$ is increasing, where d^S denotes the distance function in the corresponding sphere $S_t(x) := \{y \in M | d(x,y) = t\}$. We define the *Tits metric* on $M(\infty)$ by

$$Td(z,w) := \lim_{t\to\infty} (1/t)\, d^S(c_1(t), c_2(t))$$

(cf. [**3**, p. 43]).

REMARK 3.2. The definition of the Tits metric does not depend on $x \in M$ chosen. $Td\colon M(\infty) \times M(\infty) \to [0,\infty) \cup \{\infty\}$ is a metric (i.e. $d(z,w) = 0 \Leftrightarrow z = w$, $d(z,w) = d(w,z)$, $d(z,v) + d(v,w) \geq d(z,w)$). We allow that points have infinite distance.

LEMMA 3.3 [**3**, p. 46]. *Let $z, w \in M(\infty)$. If $Td(z,w) > \pi$, then there exists a unit speed geodesic $c\colon R \to M$ such that $c(-\infty) = z$, $c(\infty) = w$.*

LEMMA 3.4 [**3**, p. 46]. *If $c\colon R \to M$ is a unit speed geodesic, then*

$$Td(c(-\infty), c(\infty)) \geq \pi.$$

Equality holds if and only if c bounds a totally geodesic flat half plane.

DEFINITION 3.5. Let $c\colon [0,1] \to M(\infty)$ be a continuous curve in the Tits-topology (i.e. the topology defined by the Tits metric). We define the *length* of c by $L(c) := \sup \sum_{i=0}^{m-1} Td(c(t_i), c(t_{i+1}))$, where $0 = t_0 \leq t_1 \leq \cdots \leq t_m = 1$ is a subdivision of $[0,1]$ and the supremum is taken over all subdivisions.

LEMMA 3.6 [**3**, p. 49]. *Let $z, w \in M(\infty)$. If $Td(z,w) < \infty$, then there exists a continuous curve $h\colon [0,1] \to M(\infty)$ in the Tits-topology such that $h(0) = z$, $h(1) = w$ and $L(h|_{[0,t]}) = t\,Td(z,w)$ for any $t \in [0,1]$. (h is called a minimal geodesic in $M(\infty)$ from z to w.)*

4. The Visibility Axiom.

LEMMA 4.1 [**3**, p. 54]. *Let M be a Hadamard manifold. M satisfies the Visibility Axiom if and only if for every different points $z, w \in M(\infty)$ there is a unit speed geodesic $c\colon R \to M$ with $c(-\infty) = z$, $c(\infty) = w$.*

For its proof, see the reference above.

PROPOSITION 4.2 [**9**, p. 488]. *Let M be a Hadamard manifold whose sectional curvature K satisfies $-k^2 \leq K \leq 0$ for some constant $k > 0$. Let H be an arbitrary horosphere of M. Then for any points $p, q \in H$ we have*

$$d^H(p,q) \leq \frac{2}{k} \sinh \frac{k}{2} d(p,q),$$

where d^H is the distance function in the corresponding horosphere.

For its proof, see the reference above.

THEOREM 4.3. *Let M be a Hadamard manifold whose sectional curvature K satisfies $-k^2 < K \leq 0$ for some constant $k > 0$, and assume that the geodesic flow on SM is of Anosov type. Then for any different points $z, w \in M(\infty)$, there exists a unit speed geodesic c which satisfies $c(-\infty) = z$, $c(\infty) = w$.*

PROOF. For any different points z, $w \in M(\infty)$, we prove $Td(z, w) = \infty$. Then this theorem follows from Lemma 3.3. Assume that $Td(z, w) < \infty$ for some $z, w \in M(\infty)$, $z \neq w$. From Lemma 3.6, there exists a minimal geodesic $h\colon [0,1] \to M(\infty)$ from z to w. Let $\gamma\colon R \to M$ be a unit speed geodesic with $\gamma(\infty) = z$, and let $a_1 := Td(\gamma(-\infty), \gamma(\infty))$. From Lemma 3.4, $a_1 \geq \pi$. If $a_1 = \pi$, then again from Lemma 3.4 γ bounds a totally geodesic flat half plane. Then using the variation of γ through geodesics, we obtain a nonzero perpendicular Jacobi field Y along γ such that $\|Y(t)\| = 1$ for all $t \in R$. This contradicts Proposition 2.10, and it follows that $a_1 > \pi$. Since $h(0) = z$, there exists a number $s \in (0, 1)$ with

$$(*) \qquad Td(z, h(s)) < \min\{2c/k, a_1 - \pi\}.$$

Let $v := h(s)$. Since $Td(\gamma(-\infty), v) \geq Td(\gamma(-\infty), \gamma(\infty)) - Td(\gamma(\infty), v) > a_1 - (a_1 - \pi) = \pi$, from Lemma 3.3 there exists a unit speed geodesic $\mu\colon R \to M$ such that $\mu(-\infty) = \gamma(-\infty)$, $\mu(\infty) = v$ and $\gamma(t)$, $\mu(t)$ lie on the same horosphere with center $\gamma(-\infty)$ for any $t \in R$. Let $\mu_0\colon [0, \infty) \to M$ be the unique unit speed geodesic with $\mu_0(0) = \gamma(0)$, $\mu_0(\infty) = v$. Since μ and μ_0 are asymptotic, there exists a number $b_1 > 0$ such that

$$(**) \qquad d(\mu(t), \mu_0(t)) \leq b_1 \quad \text{for any } t \geq 0.$$

From Lemma 2.8 and Proposition 4.2, for any $t \geq 0$

$$\begin{aligned} b(1+k^2)^{-1/2} e^{ct} d^H(\gamma(0), \mu(0)) &\leq d^H(\gamma(t), \mu(t)) \\ &\leq (2/k)\sinh((k/2) d(\gamma(t), \mu(t))) \\ &< (1/k)\exp((k/2) d(\gamma(t), \mu(t))). \end{aligned}$$

Then,

$$bk(1+k^2)^{-1/2} e^{ct} d^H(\gamma(0), \mu(0)) < \exp((k/2) d(\gamma(t), \mu(t))).$$

Take the logarithm of both sides and divide by $kt/2$ to obtain

$$(***) \qquad 2c/k + (2/kt)B < (1/t) d(\gamma(t), \mu(t)),$$

where $B = \log\{bk(1+k^2)^{-1/2} d^H(\gamma(0), \mu(0))\}$.

Then for any $\varepsilon > 0$ there exists a number $N > 0$ such that $2c/k - \varepsilon/2 < (1/t) d(\gamma(t), \mu(t))$ for any $t \geq N$. Choose a number $t_1 \geq N$ such that $b_1/t_1 < \varepsilon/2$. From $(**)$, we have

$$\begin{aligned} Td(z, v) &= \lim_{t\to\infty} (1/t) d^S(\gamma(t), \mu_0(t)) \\ &\geq (1/t_1) d^S(\gamma(t_1), \mu_0(t_1)) \\ &\geq (1/t_1) d(\gamma(t_1), \mu_0(t_1)) \\ &\geq (1/t_1)\{d(\gamma(t_1), \mu(t_1)) - b_1\} \\ &> (2c/k - \varepsilon/2) - \varepsilon/2 = 2c/k - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $Td(z, v) \geq 2c/k$. But this contradicts $(*)$.

COROLLARY 4.4. *Let M satisfy the hypotheses of Theorem* 4.3. *Then M satisfies the Visibility Axiom.*

PROOF. This corollary follows from Theorem 4.3 and Lemma 4.1.

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