THE ISOMETRY GROUPS OF COMPACT MANIFOLDS WITH NEGATIVE RICCI CURVATURE

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ABSTRACT. We estimate the order of the isometry groups of compact manifolds with negative Ricci curvature in terms of geometric quantities: the sectional curvature, the Ricci curvature, the diameter, and the injectivity radius.

1. Introduction. Let M be a compact Riemannian manifold with negative Ricci curvature. Bochner [2] showed that there is no nontrivial Killing vector field on M and therefore the isometry group I(M) of M has finite order. The purpose of this note is to estimate the order of I(M). In this direction, when M is nonpositively curved, Huber [6], Im Hof [7] and Maeda [8] obtained similar results. Moreover, there are some generalizations by Yamaguchi [9], Adachi and Sunada [1].

Let $K_M$ be the sectional curvature, $\text{Ric}_M$ the Ricci curvature, $D_M$ the diameter, and $i_M$ the injectivity radius of M, respectively.

THEOREM. Given an integer $n$ and positive constants $\Lambda, c, D, i$, there is a constant $N$ depending only on $n, \Lambda, c, D, i$ such that if a compact connected $n$-dimensional Riemannian manifold M satisfies $|K_M| < \Lambda^2$, $\text{Ric}_M \leq -c$, $D_M \leq D$, $i_M \geq i$, then the order of the isometry group I(M) is smaller than $N$.

It should be noted that the constant $N$ depends very essentially on the bound of the sectional curvature and it is an interesting problem to decide whether or not this dependence is essential.

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2. Proof. The following Lemma is crucial in the proof of the Theorem. It is a discrete version of Bochner’s classical argument for the nonexistence of Killing vector fields. Let $d(\cdot, \cdot)$ be the distance on M induced by the Riemannian metric. For the sake of brevity, we normalize $\Lambda = 1$.

LEMMA. If an isometry $\phi$ of M satisfies $d(p, \phi(p)) < \min(i, \pi/4, c/(n - 1))$ for all $p$ in M, then $\phi$ is the identity map.

PROOF. Let $p$ be the point such that $d(p, \phi(p))$ is maximal and let $\gamma: [0, t_0] \to M$ be the unit speed geodesic from $p$ to $\phi(p)$. Note that $\gamma$ is an axis of $\phi$. Let $H_0 = \exp_p(U)$ for a sufficiently small neighborhood $U$ of 0 in the normal space of $\gamma$.
at p. Now use Gromov’s approach to the Rauch comparison theorem [5, 8.9]. Let $H_t$ be equidistant local hypersurfaces along $\gamma$, $S = S_t$ the second fundamental form of $H_t$ at $\gamma(t)$ in the direction $-\dot{\gamma}(t)$, $S_0 = 0$. Then, since $\phi$ is an isometry, $\phi(H_0)$ is totally geodesic at $\phi(p)$, touches $H_{t_0}$ at $\phi(p)$, but lies completely on the “left” of $H_{t_0}$ because $\phi$ has maximal displacement at $p$. This is impossible certainly if

\[ (*) \quad \text{tr } S_{t_0} > 0. \]

Now $S' + S^2 + R_\gamma = 0$, $R_\gamma T = R(T, \dot{\gamma})\dot{\gamma}$, and the assumption $-1 \leq K_M \leq 1$ yield immediately

\[ -\tan t \leq S \leq \tanh t. \]

Thus, $\text{tr } S^2 \leq (n - 1)\tan^2 t$, and ($*$) holds for

\[ (n - 1)\tan^2 t_0 < (n - 1)\tan t_0 < c, \]

if $1 - n \leq \text{Ric}_M \leq -c$ and $t_0 < \pi/4$. Q.E.D.

Now we prove the Theorem. The method is fairly standard by now (cf. [8]). As usual, $B_r(p)$ will denote the closed ball of radius $r$ with center $p$.

Take $a < \min(i/4, \pi/16, c/4(n-1))$ and $\{p_i\}_{i=1}^l$ such that $M \subset \bigcup_{i=1}^l B_a(p_i)$. We define the map $F$ from $I(M)$ to the symmetric group $S_l$ of degree $l$ by $F(\phi): i \mapsto j(i)$, where $j(i)$ is the smallest $j$ such that $\phi(p_i) \in B_a(p_j)$. We will show $F$ is injective. For this assume $F(\phi) = F(\psi) = j(\cdot)$. Take an arbitrary point $p$ and say $p \in B_a(p_i)$. Then,

\[ d(\phi(p), \psi(p)) \leq d(\phi(p), \phi(p_i)) + d(\phi(p_i), \phi(p_{j(i)})) + d(\phi(p_{j(i)}), \psi(p_i)) + d(\psi(p_i), \psi(p)) \leq 4a. \]

Thus the above Lemma shows $\phi = \psi$.

On the other hand, by the volume comparison theorem [5], $l$ is smaller than $b(D)/b(a/2)$ where $b(t)$ is the volume of the closed ball of radius $t$ in the space with constant curvature $-1$. Hence the order of $I(M)$ is smaller than that of $S_l$. Q.E.D.

REFERENCES


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