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THE ISOMETRY GROUPS OF COMPACT MANIFOLDS WITH NEGATIVE RICCI CURVATURE

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ABSTRACT. We estimate the order of the isometry groups of compact manifolds with negative Ricci curvature in terms of geometric quantities: the sectional curvature, the Ricci curvature, the diameter, and the injectivity radius.

1. Introduction. Let M be a compact Riemannian manifold with negative Ricci curvature. Bochner [**2**] showed that there is no nontrivial Killing vector field on M and therefore the isometry group $I(M)$ of M has finite order. The purpose of this note is to estimate the order of $I(M)$. In this direction, when M is nonpositively curved, Huber [**6**], Im Hof [**7**] and Maeda [**8**] obtained similar results. Moreover, there are some generalizations by Yamaguchi [**9**], Adachi and Sunada [**1**].

Let K_M be the sectional curvature, Ric_M the Ricci curvature, D_M the diameter, and i_M the injectivity radius of M, respectively.

THEOREM. *Given an integer n and positive constants Λ, c, D, i, there is a constant N depending only on n, Λ, c, D, i such that if a compact connected n-dimensional Riemannian manifold M satisfies $|K_M| \leq \Lambda^2$, $\mathrm{Ric}_M \leq -c$, $D_M \leq D$, $i_M \geq i$, then the order of the isometry group $I(M)$ is smaller than N.*

It should be noted that the constant N depends very essentially on the bound of the sectional curvature and it is an interesting problem to decide whether or not this dependence is essential.

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2. Proof. The following Lemma is crucial in the proof of the Theorem. It is a discrete version of Bochner's classical argument for the nonexistence of Killing vector fields. Let $d(\cdot,\cdot)$ be the distance on M induced by the Riemannian metric. For the sake of brevity, we normalize $\Lambda = 1$.

LEMMA. *If an isometry ϕ of M satisfies $d(p, \phi(p)) < \min(i, \pi/4, c/(n-1))$ for all p in M, then ϕ is the identity map.*

PROOF. Let p be the point such that $d(p, \phi(p))$ is maximal and let $\gamma\colon [0, t_0] \to M$ be the unit speed geodesic from p to $\phi(p)$. Note that γ is an axis of ϕ. Let $H_0 = \exp_p(U)$ for a sufficiently small neighborhood U of 0 in the normal space of γ

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at p. Now use Gromov's approach to the Rauch comparison theorem [**5**, 8.9]. Let H_t be equidistant local hypersurfaces along γ, $S = S_t$ the second fundamental form of H_t at $\gamma(t)$ in the direction $-\dot{\gamma}(t)$, $S_0 = 0$. Then, since ϕ is an isometry, $\phi(H_0)$ is totally geodesic at $\phi(p)$, touches H_{t_0} at $\phi(p)$, but lies completely on the "left" of H_{t_0} because ϕ has maximal displacement at p. This is impossible certainly if

$$(*) \qquad \operatorname{tr} S_{t_0} > 0.$$

Now $S' + S^2 + R_{\dot{\gamma}} = 0$, $R_{\dot{\gamma}}T = R(T, \dot{\gamma})\dot{\gamma}$, and the assumption $-1 \leq K_M \leq 1$ yield immediately

$$-\tan t \leq S \leq \tanh t.$$

Thus, $\operatorname{tr} S^2 \leq (n-1)\tan^2 t$, and $(*)$ holds for

$$(n-1)\tan^2 t_0 < (n-1)t_0 < c,$$

if $1 - n \leq \mathrm{Ric}_M \leq -c$ and $t_0 < \pi/4$. Q.E.D.

Now we prove the Theorem. The method is fairly standard by now (cf. [**8**]). As usual, $B_r(p)$ will denote the closed ball of radius r with center p.

Take $a < \min(i/4, \pi/16, c/4(n-1))$ and $\{p_i\}_{i=1}^l$ such that $M \subset \bigcup_{i=1}^l B_a(p_i)$. We define the map F from $I(M)$ to the symmetric group S_l of degree l by $F(\phi)\colon i \to j(i)$, where $j(i)$ is the smallest j such that $\phi(p_i) \in B_a(p_j)$. We will show F is injective. For this assume $F(\phi) = F(\psi) = j(\cdot)$. Take an arbitrary point p and say $p \in B_a(p_i)$. Then,

$$\begin{aligned} d(\phi(p), \psi(p)) &\leq d(\phi(p), \phi(p_i)) + d(\phi(p_i), p_{j(i)}) \\ &\quad + d(p_{j(i)}, \psi(p_i)) + d(\psi(p_i), \psi(p)) \\ &\leq 4a. \end{aligned}$$

Thus the above Lemma shows $\phi = \psi$.

On the other hand, by the volume comparison theorem [**5**], l is smaller than $b(D)/b(a/2)$ where $b(t)$ is the volume of the closed ball of radius t in the space with constant curvature -1. Hence the order of $I(M)$ is smaller than that of S_l. Q.E.D.

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