ABSTRACT. A map into the Hilbert cube is stable if each composition with projection onto a finite number of factors is stable. We prove that a map from a compact metric space into the Hilbert cube is stable if and only if it is universal. As a consequence, the composition of a stable map with any self homeomorphism of the Hilbert cube is also stable.

1. Introduction. All spaces are separable metric spaces. A map \( f: X \to I^n \) is said to be stable if there does not exist a map \( g: X \to S^{n-1} \) with \( f|_{f^{-1}(S^{n-1})} = g|_{f^{-1}(S^{n-1})} \). Stable maps are also known as Alexander-Hopf essential maps [N, GT]. Krasinkiewicz has given a general definition of essential maps into the product of manifolds [K] that coincides with the definition of stable maps in the cases under consideration. It is well known that a space has covering dimension greater than or equal to \( n \) if and only if it admits a stable map into \( I^n \) [HW]. Note that if \( f: X \to I^n \) is a stable map and \( h \) is any self-homeomorphism of \( I^n \), then the composition \( h \circ f \) is also stable, since \( h(S^{n-1}) = S^{n-1} \).

Let \( I^\infty = \prod_{i=1}^\infty [-1, 1] \) denote the Hilbert cube, and for each \( n \) let \( p_n: I^\infty \to I^n \) be the projection map onto the first \( n \) factors. The subsets \( A_n = \{(x_i) \in I^\infty | x_n = -1\} \) and \( B_n = \{(x_i) \in I^\infty | x_n = 1\} \) are referred to as faces of the Hilbert cube. A map \( f: X \to I^\infty \) from a compact metric space into the Hilbert cube is said to be stable if the composition \( p_n \circ f: X \to I^n \) is stable for each \( n \). See [W and B] for a more detailed description of stable maps. In particular, Walsh showed that a map \( f: X \to I^\infty \) from a compact metric space is stable if and only if the collection of pairs \( \{(f^{-1}(A_i), f^{-1}(B_i)) | i = 1, 2, \ldots \} \) is an essential family for \( X \), i.e., for any sequence of separators \( \{S_i\} \) of \( f^{-1}(A_i) \) and \( f^{-1}(B_i) \), \( \bigcap_{i=1}^\infty S_i \neq \emptyset \). It follows that a compact metric space admits a stable map into the Hilbert cube if and only if it is strongly infinite dimensional.

Our goal is to prove a result for stable maps into the Hilbert cube which is analogous to the result noted above for stable maps into \( n \)-cells. Namely, we show that if \( f: X \to I^\infty \) is stable and \( h \) is any self-homeomorphism of \( I^\infty \), then the composition \( h \circ f \) is also stable. This gives a partial answer to a question of J. Krasinkiewicz [K, Problem 1]. What we need for the proof is a characterization of stable maps in terms of a property preserved by self-homeomorphisms. Universality, a concept introduced by Holsztyński [H1] and widely used in the study of fixed point theory (see [H2, H3, H4, H5, H6, H7, GT and N]), is such a property. A map

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f: X → Y is said to be universal if for every map g: X → Y there exists a point p in X with f(p) = g(p). We will show that a map from a compact metric space into the Hilbert cube is stable if and only if it is universal. Then the desired result on preservation of stability by compositions with self-homeomorphisms of I^∞ is an immediate corollary.

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2. Universal maps and stable maps. The following result is contained in [GT, N] and implicitly in [H1]. For completeness, we include a short proof.

**Theorem 1.** Let n be in Z_. A map f: X → In is stable if and only if it is universal.

**Proof.** Suppose that f: X → In is a stable map. If f were not universal, then we could find a map g: X → Sn so that g(p) ≠ f(p) for every point p in X. Consider Sn as the boundary of In in the usual manner, and define h: X → Sn by setting h(p) equal to the intersection of the ray containing f(p) which emanates from g(p) and Sn. Clearly h is continuous and agrees with f on f^(-1)(Sn), contradicting the stability of f. Therefore, f must be universal.

For the converse, suppose that f: X → In is universal and again consider Sn as the boundary of In. If f were not stable, then we could find a map g: X → Sn which agrees with f on f^(-1)(Sn). Composing with the antipodal map α: Sn → Sn would then give a map αo: X → Sn ⊆ In so that αo(g(p)) ≠ f(p) for any point p in X, contradicting the universality of f. Thus, f must be stable. □

The next result is the tool needed to link stability and universality of maps of compacta into the Hilbert cube.

**Theorem 2.** A map f: X → I^∞ from a compact metric space into the Hilbert cube is universal if and only if each composition pn o f: X → In is universal.

**Proof.** Assume that f is universal. Fix a positive integer n. Consider the Hilbert cube as In × ∏j>n [-1,1]j, and let a map g: X → In be given. By choosing a point y_j in [-1,1]_j for each j > n, we may assume that g is a map into the Hilbert cube. Thus, since f is universal, there exist a point p in X so that g(p) × ∏j>n {y_j} = f(p). Thus, g(p) = pn o f(p), and pn o f is shown to be universal.

For the converse, assume that the Hilbert cube has the metric given by d(y, y') = ∑i=1 In (|y_i - y'_i|/2^i). Suppose that pn o f is universal for each positive integer n. If f were not universal, then there exists a map g: X → I^∞ so that for every point p in X, g(p) ≠ f(p). By the compactness of X, there exists a number δ > 0 so that for every point p in X the distance d(g(p), f(p)) > δ. But we can choose a positive integer N so that the diam(∏i≥n+1 [-1,1]_i) < δ. This would imply that, for any point p in X, pn o g(p) ≠ pn o f(p) contradicting the universality of pn o f(p). Therefore, f must be universal. □

We are now ready to obtain the results mentioned at the end of the previous section.

**Corollary 1.** A map f: X → I^∞ from a compact metric space into the Hilbert cube is stable if and only if it is universal.

**Proof.** This is immediate from Theorems 1 and 2. □
COROLLARY 2. If \( f : X \to I^\infty \) is stable and \( h \) is a self-homeomorphism of \( I^\infty \), then \( h \circ f \) is also stable.

PROOF. Clearly, universality of maps is preserved by composition with self-homeomorphisms of \( I^\infty \). By Corollary 1, the same is true for stable maps. \( \square \)

COROLLARY 3. Let \( \{ (C_i, D_i) \}_{i=1}^{\infty} \) be an essential family for a strongly infinite-dimensional compact space \( X \), and let \( f \) be a map from \( X \) into \( I^\infty \) so that \( C_i = f^{-1}(A_i) \) and so that \( D_i = f^{-1}(B_i) \) for each \( i \). If \( h \) is any self-homeomorphism of \( I^\infty \), then \( \{ (h \circ f)^{-1}(A_i), (h \circ f)^{-1}(B_i) \}_{i=1}^{\infty} \) is also an essential family for \( X \).

PROOF. This follows immediately from Corollary 2 and Walsh’s characterization of stability mentioned in §1. \( \square \)

REFERENCES


