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POINTWISE ESTIMATES FOR THE RELATIVE FUNDAMENTAL SOLUTION OF $\overline{\partial}_b$

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ABSTRACT. Consider a compact pseudoconvex CR manifold of dimension 3 and finite type, on which the operator $\overline{\partial}_b$ has closed range in L^2. The relative fundamental solution of $\overline{\partial}_b$ is the distribution-kernel for that operator which inverts $\overline{\partial}_b$, modulo its kernel and cokernel. We derive pointwise bounds on this fundamental solution and its derivatives.

Let M be a compact CR manifold of real dimension 3. We assume that M is pseudoconvex and of finite type m, and that the $\overline{\partial}_b$ operator on M has closed range on L^2. The latter holds automatically when M is the boundary of a smooth, bounded pseudoconvex domain in C^2. Fix a positive measure on M with a smooth, nonvanishing density in local coordinates. Let S denote the Szegö projection of L^2, with respect to this measure, onto the kernel H_b of $\overline{\partial}_b$ in L^2. $\overline{\partial}_b$ maps (test) functions to sections of a bundle $B^{0,1}$; fix an inner product structure on the bundle and let L^{2*} denote the Hilbert space of L^2 sections of $B^{0,1}$. Let $\overline{\partial}_{b^*}$ denote the adjoint operator, let S^* denote the orthogonal projection of L^{2*} onto the kernel $H_{b^*} \subset L^{2*}$ of $\overline{\partial}_{b^*}$, and let K, K^* be the distribution-kernels for S, S^* respectively. For definitions of all these terms and references see for instance [**C**], [**FK**], [**K**].

The hypothesis that $\overline{\partial}_b$ has closed range means that $\mathrm{Range}(\overline{\partial}_b) = L^{2*} \cap \overline{\partial}_b(L^2)$ is a closed subspace of L^{2*}, and that for each $f \in \mathrm{Range}(\overline{\partial}_b)$ there exists a unique $u \in L^2$ satisfying

$$\begin{cases} \overline{\partial}_b u = f, \\ u_b \perp H_b. \end{cases}$$

Moreover $\|u\|_2 \leq C\|f\|_2$. Therefore the operator G which maps any $f \in L^{2*}$ to the unique $u \perp H_b$ satisfying $\overline{\partial}_b u = (I - S^*)f$, is bounded from L^{2*} to L^2. Let L denote its distribution-kernel, the relative fundamental solution for $\overline{\partial}_b$. The purpose of this article is to obtain certain pointwise bounds for L and its derivatives. This is a continuation of the work [**C**] and is based on the results obtained there; we shall continue to employ the notation of that paper without full explanation. In particular the bounds we seek for L are formulated in terms of a quasi-metric ρ and a family of balls $B(x,r)$ on M, constructed and studied in the fundamental paper [**NSW**], which are induced by the CR structure in a natural way. In this connection $\hat{B}$ denotes the unit ball in $\mathbf{R}^3$, and for each $x \in M$, $r \in (0, C_M]$ there is given a special coordinate map $\phi_{x,r}$, a diffeomorphism of $\hat{B}$ onto $B(x,r)$. $\Lambda(x,r)$

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denotes the measure of $B(x,r)$. For a summary of their relevant properties see section 15 of [**C**].

In local coordinates in M, $\overline{\partial}_b$ takes the form $X+iY$ where X,Y are real, smooth vector fields, linearly independent at every point. Define, for x,y in a coordinate patch, $\vartheta(x,y)$ to be the infimum of all r such that there exists an absolutely continuous function ψ from $[0,r]$ into the coordinate patch, with $\psi(0)=x$ and $\psi(r)=y$, such that for almost all t, $d\psi/dt = a(t)X(\psi(t))+b(t)Y(\psi(t))$, with $a^2(t)+b^2(t)\le 1$. Then $\vartheta(x,y)$ is finite, and there is a uniform inequality $C\rho(x,y)\le\vartheta(x,y)\le C'\rho(x,y)$. $B(x,r)$ is $\{y\colon \rho(x,y)<r\}$. Equivalent reformulations of the estimates below may be obtained by replacing ρ by ϑ and $B(x,r)$ by $\{y\colon \vartheta(x,y)<r\}$; the measures of $B(x,r)$ and $\{y\colon \vartheta(x,y)<r\}$ are comparable, uniformly in x and r.

We denote by D any differential operator of the form $(X \text{ or } Y)\circ(X \text{ or } Y)\ldots$ and let n be the number of factors of $(X$ or $Y)$, possibly zero. D_x denotes such an operator acting in the x-variable, with n factors, and D_y acts in the y-variable and has n' factors.

Our main result is

THEOREM 1. *L is C^∞ away from the diagonal*

$$|D_xD_yL(x,y)|\le C_{n,n'}r^{1-n-n'}\Lambda(x,r)^{-1}$$

uniformly for all n,n' and $x\ne y\in M$, where $r=\rho(x,y)$.

An immediate consequence is

THEOREM 2. *Suppose that $f\in L^2*$, $u\perp H_b$ and $\overline{\partial}_b u=f$. Suppose further that f is bounded on some open set $U\subset M$. Then u is Hölder continuous of order m^{-1} on every compact subset of U.*

This follows directly from the first theorem, by Theorem 14(b) of [**RS**]. Moreover Theorem 1 implies that $|u(x)-u(y)|\le C\rho(x,y)\log(\rho(x,y)^{-1})$ as $\rho(x,y)$ tends to 0. (Recall that $\rho(x,y)\le C|x-y|^\delta$, where $\delta=m^{-1}$ [**NSW**].) Under the hypothesis of type m, this is the best order of regularity that could be concluded, even if it were known that Xu, Yu were separately bounded on U. Thus the results of Theorem 1 are fairly sharp. Theorem 2 has also been obtained by Fefferman and Kohn [**FK**].

To begin the proofs observe that L is C^∞ away from the diagonal. For the distribution-kernel for $I-S^*$ is C^∞ off of the diagonal (see below), and the solution $u\perp H_b$ of $\overline{\partial}_b u=h$ is C^∞ wherever h is. See [**K**] (or [**C**]). It remains to examine L near the diagonal.

Consider any distinct points x_0,y_0 in a common coordinate patch, close together. let $c_1\ll 1\ll c_2$ be two constants depending only on M, very small and very large respectively. Let $r=\rho(x_0,y_0)$, $B=B(x_0,c_2r)$, $B_1=B(y_0,c_1r)$, $B_3=B(y_0,2c_1r)$, $B_2=B(x_0,c_1r)$, and $B_4=B(x_0,2c_1r)$. To analyze L and its x-derivatives at (x_0,y_0) we consider the map from $L^{2*}(B_1)$ to $C^\infty(B_2)$ which sends any $f\in L^{2*}$ supported on B_1 to Gf restricted to B_2. Let $u=Gf$.

The first step is to analyze $(I-S^*)f$. In [**C**] was proved

THEOREM A. *K^* is C^∞ away from the diagonal and satisfies*

$$|D_xD_yK^*(x,y)|\le C_{n,n'}\rho(x,y)^{-n-n'}\Lambda(x,\rho(x,y))^{-1}$$

for all D_x, D_y. The same holds for K.

Let $h = (I - S^*)f$ and $\hat{h} = h \circ \phi_{x_0,c_2r}$ on $\hat{B}$. Let $\hat{X}$ and $\hat{Y}$ be the pullbacks of X, Y and let $\hat{D} = (\hat{X} \text{ or } \hat{Y}) \circ (\hat{X} \text{ or } \hat{Y}) \ldots$ with n factors on $\hat{B}$. From Theorem A and the restriction that f be supported on B_1 there easily follows

$$\|Dh\|_{L^2(B_2)} \leq C_n r^{-n} \|f\|_2.$$

Equivalently

COROLLARY 3.

$$\|\hat{D}\hat{h}\|_{L^2(\hat{B}_2)} \leq C_n \Lambda(x_0, r)^{-1/2} \|f\|_2$$

for all $n \geq 0$.

For each x_0, y_0 there exists $\hat{\psi} \in C_0^\infty(\hat{B})$ with C^k norm bounded uniformly in x_0, y_0 for all k, satisfying $\hat{\psi} \equiv 1$ on $\phi^{-1}(B_4)$ and $\hat{\psi} \equiv 0$ on $\phi^{-1}(B_3)$, where $\phi = \phi_{x_0,c_2r}$. Let $\psi = \hat{\psi} \circ \phi^{-1}$, which may be viewed as a C^∞ function on M supported on B_4 by virtue of the compact support of $\hat{\psi}$. Let $v \in L^{2*}$ be the unique solution of

$$\begin{cases} \overline{\partial}_{b^*} v = (I - S)(\psi u), \\ v \perp H_{b^*}. \end{cases}$$

$I - S$ projects onto Range$(\overline{\partial}_{b^*})$, the orthocomplement of H_b, so a solution exists.

LEMMA 4.

$$\|\psi u\|_2 \leq Cr\|f\|_2$$

and

$$\|v\|_{L^2(B)} \leq Cr^2 \|f\|_2$$

uniformly for all x_0, y_0, f.

This is an immediate consequence of

PROPOSITION B [**C**]. *If* $f \in \text{Range}(\overline{\partial}_b)$ *and* $u \perp H_{b^*}$ *satisfies* $\overline{\partial}_b u = f$ *then*

$$\|u\|_{L^2(B(x,r))} \leq Cr\|f\|_2$$

uniformly for all $x \in M$, $r > 0$, f. *The corresponding estimate is valid for the* $\overline{\partial}_{b^*}$ *equation.*

v is introduced in order to obtain the factor of r^2 in Lemma 4, which permits the rescaling argument below. It would be more natural to consider the solution w of $\overline{\partial}_{b^*} w = u$ with $w \perp H_{b^*}$, but we do not know that $\|w\|_{L^2(B)} \leq Cr^2\|f\|_2$. Otherwise Theorem 1 would be an immediate consequence of the arguments in [**C**].

Restrict everything to B_4, and let $z = (I - S)(\psi u)$. Then

$$\begin{cases} \overline{\partial}_{b^*} v = z, \\ \overline{\partial}_b z = h, \end{cases}$$

with the $L^2(B_4)$ norms satisfying, for all n,

$$\|r^{-2} v\|_2 + \|r^{-1} z\|_2 + \|r^n Dh\|_2 \leq C_n \|f\|_{L^2(M)}.$$

Now pull everything back to $\hat{B}$ via $\phi = \phi_{x_0, 2c_1 r}$. Let $\hat{g} = g \circ \phi$ for any g defined on B, and let $\hat{\partial}$ and $\hat{\partial}_*$ be the pullbacks of $\overline{\partial}_b$ and $\overline{\partial}_{b^*}$, respectively. Then the equations rescale to

$$\begin{cases} \hat{\partial}_*(r^{-2}\hat{v}) = (r^{-1}\hat{z}), \\ \hat{\partial}(r^{-1}\hat{z}) = \hat{h}, \end{cases} \tag{1}$$

with the control

$$\|r^{-2}\hat{v}\|_2 + \|r^{-1}\hat{z}\|_2 + \|\hat{D}\hat{h}\|_2 \le C_n\Lambda(x_0,r)^{-1/2}\|f\|_2 \tag{2}$$

for all n, uniformly in x_0, y_0.

It is proved in [**K**] (see also [**C**]) that this implies

$$\|r^{-1}\hat{z}\|_{C^k} \le C_k\Lambda(x_0,r)^{-1/2}\|f\|_2$$

on any fixed compact subset of $\hat{B}$, for all k. Therefore on the inverse image of B_2

$$\|\hat{D}r^{-1}\hat{z}\|_{L^\infty(\hat{B})} \le C_n\Lambda(x_0,r)^{-1/2}\|f\|_2,$$

which is to say that

$$\|D[(I-S)(\psi u)]\|_{L^\infty(B_2)} \le C_n r^{1-n}\Lambda(x_0,r)^{-1/2}\|f\|_2 \tag{3}$$

for all D.

$D[u-(I-S)(\psi u)]$ may be estimated more directly, on B_2. Let u_j be the restriction of u to $B(x_0,2^jr)\setminus\bigcup_{i<j}B(x_0,2^ir)$. $u=(I-S)u$ since $u\perp H_b$, so $[u-(I-S)(\psi u)]=(I-S)[(1-\psi)u]$. Hence on B_2

$$[u-(I-S)(\psi u)] = -\sum_{j=0}^{\infty} Su_j.$$

Fix any D_x and let K_j be the restriction of D_xK to $\{(x,y)\colon x\in B_2 \text{ and } \rho(x,y)\sim 2^jr\}$ so that $D_xSu_j=\int K_j(x,y)u_j(y)\,dy$ on B_2. Then

$$\begin{aligned}\|D_xSu_j\|_{L^\infty(B_2)} &\le C\sup_x\|K_j(x,\cdot)\|_\infty\|u_j\|_1\\ &\le C(2^jr)^{-n}\Lambda(x_0,2^jr)^{-1}\|u_j\|_2\Lambda(x_0,2^jr)^{1/2}\\ &\le C(2^jr)^{1-n}\Lambda(x_0,2^jr)^{-1/2}\|f\|_2.\end{aligned}$$

We have used Theorem A to estimate K_j and Proposition B to estimate $\|u_j\|_2$, and have used the facts that ρ satisfies a quasi-triangle inequality, and that $\Lambda(x,C2^j)\approx\Lambda(x_0,2^j)$ for $x\in B_2$. $\Lambda(x_0,2^jr)\ge C2^{4j}\Lambda(x_0,r)$ [**C**, §15], so

$$\begin{aligned}\|D_x[u-(I-S)(\psi u)]\|_{L^\infty(B_2)} &\le Cr^{1-n}\Lambda(x_0,r)^{-1/2}\|f\|_2\cdot\sum_{j\ge 0}2^{j(1-n)}2^{-2j}\\ &\le Cr^{1-n}\Lambda(x_0,r)^{-1/2}\|f\|_2.\end{aligned}$$

Together with (3), since $u(x)=\int L(x,y)f(y)\,dy$, this establishes

LEMMA 5. *For all distinct $x_0,y_0\in M$ and all D_x*

$$\|D_xL(x,\cdot)\|_{L^2(B(y_0,c_1r))} \le C_nr^{1-n}\Lambda(x_0,r)^{-1/2}$$

for all $x\in B(x_0,c_1r)$ where $r=\rho(x_0,y_0)$.

The next claim is that the same holds with the roles of the variables reversed:

$$\|D_y L(\cdot, y)\|_{L^2(B(x_0,c_1r))} \le Cr^{1-n'}\Lambda(x_0,r)^{-1/2} \tag{4}$$

for all $y \in B(y_0, c_1 r)$. ($\Lambda(x_0,r) \sim \Lambda(y_0,r)$ so the lack of symmetry is only apparent.) Observe that the adjoint G^* of G is the operator which first maps any $g \in L^2$ to $(I-S)g$, then sends it to the solution $v \perp H_{b^*}$ of $\overline{\partial}_{b^*} v = (I-S)g$; in other words the distribution-kernel for G^* is the relative fundamental solution for $\overline{\partial}_{b^*}$. Since the whole machine applies equally well to $\overline{\partial}_{b^*}$ as to $\overline{\partial}_b$, (4) follows from a repetition of the proof of Lemma 5. To verify the observation note that $G^* = (I-S^*)G^*(I-S)$ since $G = (I-S)G(I-S^*)$. Thus it suffices to show that $\overline{\partial}_{b^*} \circ G^*$ is the identity on the orthocomplement of H_{b^*}. But $G\overline{\partial}_b = (I-S)$ on test functions by definition, so $\overline{\partial}_{b^*} \circ G^* = (I-S)$.

Finally pull L back to $\hat{L}$ on $\hat{B} \times \hat{B}$ via $\phi_{x_0,c_1r} \times \phi_{y_0,c_1r}$. Lemma 5 and (4) pull back to

$$\sup_{\xi} \|\hat{D}_\xi \hat{L}(\xi, \cdot)\|_2 \le C_n r\Lambda(x_0,r)^{-1}$$

and

$$\sup_{\eta} \|\hat{D}_\eta \hat{L}(\cdot, \eta)\|_2 \le C_{n'} r\Lambda(x_0,r)^{-1}$$

for all $\hat{D}_\xi, \hat{D}_\eta$, where the suprema are taken over $\hat{B}$. The additional factor of $\Lambda(x_0,r)^{-1/2}$ comes from the change of variables. Thus

$$\|(\hat{D}_\xi \text{ or } \hat{D}_\eta)\hat{L}\|_2 \le Cr\Lambda(x_0,r)^{-1/2}$$

with a bound which depends only on M and on the orders of $\hat{D}_\xi$ and $\hat{D}_\eta$. $\hat{X}$ and $\hat{Y}$ together with all their commutators of length at most m span the tangent space at each point of $\hat{B}$, and do so in a uniform way. Therefore by standard elliptic theory

$$\|\hat{L}\|_{C^n} \le C_n r\Lambda(x_0,r)^{-1}$$

on any fixed compact subset of $\hat{B} \times \hat{B}$. Passing back to $M \times M$ gives

$$|D_x D_y \hat{L}(x_0,y_0)| \le C_{n,n'} r^{1-n-n'}\Lambda(x_0,r)^{-1}$$

for all D_x, D_y as desired.

Observe that because of the relations $\overline{\partial}_b G = (I-S)$ and $\overline{\partial}_{b^*} G^* = (I-S^*)$, Theorem 1 implies Theorem A. It also implies Proposition B.

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