MORDELL-WEIL GROUPS OF GENERIC ABELIAN VARIETIES
IN THE UNITARY CASE

ALICE SILVERBERG
(Communicated by Larry J. Goldstein)

ABSTRACT. The generic fibre of a fibre system of polarized abelian varieties
with level structure, and with endomorphism structure coming from a CM-
field, is defined over the function field of the moduli space for the abelian
varieties. We prove that the points on this generic abelian variety which are
defined over the function field of the moduli space form a finite group. The
methods of proof generalize those of Mordell-Weil groups of generic abelian va-
rieties, Invent. Math. 81 (1985), 71-106, to which this paper is a sequel.

Introduction. In this paper we will consider fibre systems of polarized abelian
varieties over C characterized by having a fixed CM-field embedded in their endo-
morphism algebras. If V is the moduli space for such abelian varieties (with level
structure), and W is the fibre variety constructed in [4], then the fibre over the
generic point of V is an abelian variety defined over the function field of V. We
consider the points of this abelian variety which are defined over C(V). Our result
is

THEOREM. If dimV > 0, then the Mordell-Weil group of the generic fibre is
finite.

Here, V is isomorphic to a product of domains of the form \{r \times s complex matrices
Z | 1 - Z^tZ > 0\} modulo the action of a discrete subgroup of a unitary group. The
setting for which the theorem holds is described more precisely in §1.

The analogous theorem was proved for V a noncompact quotient of the complex
upper half plane by Shioda (Theorem 5.1 of [5]). When V is a quotient of a
Hilbert-Siegel space \(H_s^\tau\) and V is the moduli space for abelian varieties whose
endomorphism algebras contain a fixed totally real field, CM-field, totally indefinite
quaternion algebra over a totally real field, or quaternion algebra over a CM-field,
the analogous theorem was proved in [6] (see also [7]). The present paper makes
use of new methods to remove the restriction on the base variety V in the CM-field
case.

I owe many thanks to Goro Shimura for suggesting the problem and giving
guidance and advice, and to Robert Indik for the proof of Theorem 6.

NOTATION. We will write diag(A_1, \ldots, A_s) for the matrix with blocks A_1, \ldots, A_s
on the diagonal, I(\tau) for the identity matrix in GL_r(\mathbb{Z}), and \tau^\gamma for the transpose
of the matrix \gamma.
1. Background. In this section we recall the set-up of [2 and 4]. Let $K$ be a totally imaginary quadratic extension of a totally real number field $F$, let $\rho$ be complex conjugation, and let $g = [F : \mathbb{Q}]$. We may view $K$ as embedded in $\mathbb{C}^g \cong K \otimes_{\mathbb{Q}} \mathbb{R}$, and for $k = 1, \ldots, g$ we let $\tau_k$ be the projection of $K \otimes \mathbb{R}$ onto the $k$th factor. Fix an integer $m > 1$ and a matrix $T \in GL_m(K)$ with $t^T\rho = -T$. Suppose $i(T^\tau_k)$ has signature $(r_k, s_k)$ for $k = 1, \ldots, g$. By renumbering the $\tau_k$'s, and replacing $\tau_k$ by $\rho \tau_k$ if necessary, we may assume $\tau_k \geq s_k$ for all $k$, and $\tau_1 = \min_k \tau_k$. We must assume $T$ is indefinite, i.e., $\tau_1 \neq m$ (otherwise the base variety $V$ is a single point and our theorem clearly fails). For $x = (x_1, \ldots, x_g) \in \mathbb{C}^g$, let $\Phi(x) = \text{diag}(x_1I(r_1), x_1I(s_1), \ldots, x_gI(r_g), x_gI(s_g)) \in M_n(\mathbb{C})$, where $n = gm$. Fix a lattice $\mathcal{M} \subset K^m$ of rank $2n$ and so that $\text{tr}_{K/\mathbb{Q}}(\mathcal{M}^\rho \mathcal{M}^\rho) \subset \mathbb{Z}$.

We can find $W \in GL_m(K)$ so that $WT^tW^\rho$ is diagonal. Therefore by adjusting the lattice $\mathcal{M}$, we may assume $T$ is diagonal and of the form $T = \alpha \cdot \text{diag}(a_1, \ldots, a_m)$ with $\alpha \in K^\times$, $\alpha^\rho = -\alpha$, and $a_1, \ldots, a_m \in F^\times$.

Let $J_k = \text{diag}(I(r_k), -I(s_k))$, for $k = 1, \ldots, g$. There exist matrices $W_k \in GL_m(\mathbb{R})$ so that $iW_kT^\tau_k^tW_k = J_k$ for $k = 1, \ldots, g$. Further, we can choose $W_k$ in the form $P_k \cdot D_k$ where $D_k$ is diagonal and $P_k$ is a permutation matrix defining a permutation of order two, $\pi_k$, on the set $\{1, \ldots, m\}$. Also we may rearrange $a_1, \ldots, a_m$ so that $W_1$ is diagonal—then $\pi_1 = 1$.

Let $H = H(T) = \bigoplus_{k=1}^g H(r_k, s_k)$ where

$$H(r, s) = \{r \times s \text{ matrices } Z | I(r) - Z^tZ \text{ is positive hermitian} \}.$$ 

For $Z = (Z_1, \ldots, Z_g) \in H$, let

$$X_k = \begin{pmatrix} I(r_k) & Z_k \\ t^TZ_k & I(s_k) \end{pmatrix}, \quad W_k = \begin{pmatrix} u_{1,k} & \cdots & u_{m,k} \\ \bar{v}_{1,k} & \cdots & \bar{v}_{m,k} \end{pmatrix}$$

with $u_{j,k} \in \mathbb{C}^{r_k}, v_{j,k} \in \mathbb{C}^{s_k}$ for $k = 1, \ldots, g$ and let

$$R_l(Z, T) = t(t^lu_{l,1}, t^lu_{l,1}, \ldots, t^lu_{l,g}, t^lu_{l,g}) \text{ for } l = 1, \ldots, m.$$ 

For $x = (x_1, \ldots, x_m) \in (K \otimes \mathbb{R})^m$ let

$$\eta_T(x, Z) = \sum_{l=1}^m \Phi(x_l)R_l(Z, T).$$

Also fix elements $v_1, \ldots, v_s \in K^m$, let $\mathcal{E} = \{a \in K | a\mathcal{M} \subset \mathcal{M}\}$, and let $\mathcal{N} = \mathcal{M} + \sum_{j=1}^s \mathcal{E}v_j$. Let $G = \{\gamma \in GL_m(K \otimes \mathbb{R}) | t^\gamma T^\gamma = T\}$, let $G_K = G \cap GL_m(K)$, and let $\Gamma = G_K \cap \{\gamma \in M_m(K) | \mathcal{M} \gamma = \mathcal{M}, N(1 - \gamma) \subset \mathcal{M}\}$. As in [6] we may assume the $v_j$'s have been chosen so that $\Gamma \subset SL_m(K)$ and $\Gamma$ has no nonidentity elements of finite order. Let $G(r, s)$ be the unitary group $\{\gamma \in GL_m(\mathbb{C}) | t^\gamma J^\gamma = J\}$, where $J = \text{diag}(I(r), -I(s))$. Then $G(r, s)$ acts on $H(r, s)$: for $Z \in H(r, s)$, and

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(r, s)$$

with $a \in M_r(\mathbb{C})$ and $d \in M_s(\mathbb{C})$, we have $\gamma(Z) = (aZ + b)(cZ + d)^{-1}$. The map $\gamma \to (W_1^{-1}t^\gamma tW_1, \ldots, W_g^{-1}t^\gamma tW_g)$ is an isomorphism from $G$ to $\bigoplus_{k=1}^g G(r_k, s_k)$, and defines an action of $G$ on $H$.
For \( Z \in \mathbf{H} \), let \( A_Z = C^n/\eta_T(\mathcal{M}, Z) \), let \( C_Z \) be the polarization of \( A_Z \) defined by \( T \) (see (11) of [2]), let \( \theta_Z = \Phi \), and let \( t_j(Z) = \eta_T(v_j, Z) \). Write \( Q_Z = (A_Z, C_Z, \theta_Z, t_1(Z), \ldots, t_n(Z)) \). Then \( Q_Z \) is a polarized abelian variety of type \((K, \Phi, \rho, T, \mathcal{M}, v_1, \ldots, v_n)\) (see Definition 1.2 of [6]). By 3.6 of [4] there is a map \( \phi: \mathbf{H} \to \mathbf{P}^M \) inducing an embedding of \( \mathbf{H}/T \) into a projective space \( \mathbf{P}^M \) so that \( V = \phi(\mathbf{H}) \) is a nonsingular variety. Further, by 3.8 of [4], there is a fibre system \((V, W, \ldots)\) of abelian varieties so that for \( Z \in \mathbf{H} \) and \( u = \phi(Z) \), the fibre \( Q_u = (A_u, \ldots) \) is isomorphic to \( Q_Z \). The variety \( W \) is analytically isomorphic to \((H \times \mathbb{C}^n)/(r \times \mathcal{M})\), where the group law and action are defined as in 3.14 of [4] or 1.4 of [6]. The fibre over a generic point of \( V \) is an abelian variety defined over the function field \( C(V) \) of \( V \).

**Proposition 1.** The group of points over \( C(V) \) of the generic fibre is isomorphic to the group of holomorphic sections from \( V \) to \( W \).

**Proof.** Corollary 2.2 of [6]. \( \square \)

**Theorem 1.** In the situation described above, the Mordell-Weil group over \( C(V) \) of the generic fibre is finite.

When \( m = 2 \), the theorem was proved for \( \dim V > 1 \) in case IIc of the Main Theorem of [6] (see §5 of [6]), for \( \dim V = 1 \) and \( V \) compact in Case Ic of the Main Theorem of [6] (see §4.3 of [6]), and for \( \dim V = 1 \) and \( V \) not compact in Shioda's Theorem 5.1 of [5]. In what follows, we will assume \( m \geq 3 \).

**2. Reduction to lower dimensions.** Our method of proof is to pull back the fibre system to fibre systems of the same type, but with \( m = 2 \). We can then use the results for \( m = 2 \) to give a dense set of points in \( V \) whose images under sections are points of finite order, and whose fibres have CM by \( K^m \). In §3 we show this suffices to prove Theorem 1.

The method of proof here differs from §6 of [6] in one significant way: in §6.4 of [6], we pulled back the fibre system to only one fibre system over a lower dimensional base. That technique no longer suffices—it is now sometimes necessary to choose as many as \( r_1 \cdot s_1 \) different pullbacks, as can be seen by examining the case \( rT = \text{diag}(1, 1, -1), K = \mathbb{Q}(i) \).

Write \( 0 \) for the element of \( \mathbf{H} \) with zero in every entry.

**Theorem 2.** Suppose \( h: \mathbf{H} \to \mathbb{C}^n \) is a holomorphic map which induces a section from \( V \) to \( W \). Then for every \( \beta \in G_K \), \( h(\beta(0)) \) gives a point of finite order in \( A_{\beta(0)} \), and for some \( \theta_\beta \) and \( \Phi' \), \((A_{\beta(0)}, C_{\beta(0)}, \theta_\beta)\) is of type \((K^m, \Phi')\).

**Proof.** Fix positive integers: \( i \leq r_1; j \leq s_1 \). Let \( T_{ij} = \alpha \cdot \text{diag}(a_i, a_{r_1+j}) \). For \( 1 \leq k \leq g \) let \((r_k', s_k')\) be the signature of \( iT_{ij}^{-i} \). Let \( M_k = \pi_k(i), N_k = \pi_k(r_1+j) \), and write \( F_k(a, b) \) for the \( r_k \times s_k \) matrix with one in position \((a, b)\) and zeros elsewhere. Define a map \( \varepsilon(= e_{ij}) \) from \( \mathbf{H}(T_{ij}) = \bigoplus_{k=1}^g H(r_k', s_k') \to \mathbf{H} \) by \( \varepsilon(z_1, \ldots, z_g) = (Z_1, \ldots, Z_g) \) with

\[
Z_k = \begin{cases} 
  z_k \cdot F_k(M_k, N_k - r_k) & \text{if } M_k \leq r_k \text{ and } N_k > r_k, \\
  z_k \cdot F_k(N_k, M_k - r_k) & \text{if } M_k > r_k \text{ and } N_k \leq r_k, \\
  0 & \text{otherwise}.
\end{cases}
\]

**Note.** Since \((r_1', s_1') = (1, 1)\), we have \( \dim \mathbf{H}(T_{ij}) > 0 \).
For \( l = 1, \ldots, m \) define projection maps

\[ p_l(x_1, \ldots, x_{mg}) = (x_1, x_{\pi_2(l)+m}, \ldots, x_{\pi_g(l)+(g-1)m}), \]

and let \( p_{ij} = p_i \oplus p_{r_i+j} \). Observe that \( p_{ij}(\eta_T(\mathcal{C}_K^m, \varepsilon(z))) = \eta_T(\mathcal{C}_K^r, z) \). Also, let \( T_l = \alpha q_l \) and observe that \( p_l(\eta_T(\mathcal{C}_K^m, \varepsilon(z))) = \eta_{T_l}(\mathcal{C}_K, 0) \), for \( l = 1, \ldots, m \) (here, \( 0 \in \mathbf{H}(T_l) \)). Write \( (A'_z, C'_z, \theta'_z) = P'_z \) for the abelian variety determined by \( Z, T, \mathcal{C}_K^m \) (as in Theorem 1 of \([2]\)). Then for \( z \in \mathbf{H}(T_{ij}) \),

\[ A'_{\varepsilon(z)} \cong \mathbf{C}^n / \eta_T(\mathcal{C}_K^m, \varepsilon(z)) \cong \mathbf{C}^{2g} / \eta_{T_{ij}}(\mathcal{C}_K^r, z) \oplus \bigoplus_{l \neq i, r_i+j} \mathbf{C}^g / \eta_{T_l}(\mathcal{C}_K, 0). \]

Fix \( \beta \in G_K \). By 2.8 of \([2]\) (see also §6.3 of \([6]\)), \( (A_\beta(z), C_\beta(z), \theta_\beta(z)) \) is isogenous to \( P'_z \) for \( z \in \mathbf{H} \). For \( z \in \mathbf{H}(T_{ij}) \), write \( \lambda_z \) for the isogeny from \( A_{\beta \circ \varepsilon(z)} \) to \( A'_{\varepsilon(z)} \), and \( \Lambda_z \) for the associated map from \( \mathbf{C}^n \) to \( \mathbf{C}^n \). As in §6 of \([6]\), the map

\[ h_{ij}(z) = p_{ij} \circ \Lambda_z \circ h \circ \beta \circ \varepsilon(z) \]

from \( \mathbf{H}(T_{ij}) \) to \( \mathbf{C}^{2g} \) is holomorphic. We can construct a fibre system of polarized abelian varieties of type \((K, \Phi_{ij}, \rho, T_{ij}, \mathcal{C}_K^r)\), representatives for \((1/M)\mathbb{Z} / \mathbb{Z}\) with base \( \mathbf{H}(T_{ij}) / \Delta \) for \( \Delta \) the principal congruence subgroup of level \( M \) in \( \{\delta \in SL_2(\mathcal{C}_K)^0 | \delta \rho T_{ij} \delta = T_{ij} \} \), for a suitable level \( M \) so that \( \Delta \) embeds in \( \beta^{-1} \Gamma \beta \) in a natural way. Here, \( \Phi_{ij} \) is defined by \( \{(r'_k, s'_k)\}_{k=1}^g \). By Theorem 5.1 of \([5]\) and Cases Ic and IIc of the Main Theorem of \([6]\), \( h_{ij} \) is a section of finite order for this new fibre system. Thus, for every \( z \in \mathbf{H}(T_{ij}) \), \( \beta \in G_K \), \( 1 \leq i \leq r_1 \), \( 1 \leq j \leq s_1 \), \( h_{ij}(z) \) is a point of finite order in \( \mathbf{C}^{2g} / \eta_{T_{ij}}(\mathcal{C}_K^r, z) \).

Now let \( z = 0 \in \mathbf{H}(T_{ij}) \) (and then \( \varepsilon(z) = 0 \) in \( \mathbf{H} \)). Let \( E_l = \mathbf{C}^{g} / \eta_{T_l}(\mathcal{C}_K^r, 0) \) for \( 1 \leq l \leq m \). \( E_l \) has CM by \((K, \Phi_l)\), where for \( x \in K \otimes \mathbb{R} \), \( \Phi_l(x) = \text{diag}(x^{\delta_1}, \ldots, x^{\delta_g}) \) with

\[ \delta_k = \begin{cases} \tau_k & \text{if } \pi_k(l) \leq r_l, \\ \rho \tau_k & \text{if } \pi_k(l) > r_l. \end{cases} \]

Then \( A'_0 \cong \bigoplus_{l=1}^m E_l \), and \( \{\Phi_l\}_{l=1}^m \) defines a map \( \Phi': K^m \to M_n(\mathbb{C}) \) and an embedding \( \bar{\theta}_0: K^m \to \text{End}(A'_0) \otimes \mathbb{Q} \). For \( \beta \in G_K \), the map \( \lambda_0 \) defines an isogeny from \((A_{\beta(0)}, C_{\beta(0)}, \lambda_0^{-1} \bar{\theta}_0 \lambda_0) \) to \((A'_0, C'_0, \bar{\theta}_0)\). For every \( i \) and \( j \), \( h_{ij}(0) \) gives a point of finite order in \( E_i \times E_{r_i+j} \). Thus \( \Lambda_0 h(\beta(0)) \) has finite order in \( A'_0 \). Consequently, for every \( \beta \in G_K \), \( h(\beta(0)) \) has finite order in \( A_{\beta(0)} \).

### 3. Conclusion of proof of Theorem I

By Theorem 5.3 of \([4]\), the fibre system \((V, W, \ldots)\) is defined over a number field \( k_\Omega \) so that for every \( u \in V \), \( k_\Omega(u) \) is the field of moduli of the fibre \( Q_u \). For \( \beta \in G_K \), let \( \bar{\beta} = \phi(\beta(0)) \) \( \in V \). Then \( A_{\bar{\beta}} \cong A_\beta(0) \).

**Proposition 3.** Suppose \( L \) is a subfield of \( \mathbb{C} \) which is finitely generated over \( \mathbb{Q} \) and contains \( k_\Omega \). There is a constant \( B_L \) so that for every \( \beta \in G_K \), \( |A_{\bar{\beta}}(L(\bar{\beta}))|_{\text{tors}} \leq B_L \).

**Proof.** This follows immediately from Theorem 2 above, and Theorem A of §7.1 of \([6]\) (see also Theorem 2 of \([7]\)).
PROPOSITION 4. All sections are torsion.

PROOF (SEE §7.8 OF [6]). Fix a section and choose $L$ to be a field over which the section is defined. The result follows from Proposition 3 and the fact that $\{\beta | \beta \in G_K\}$ is dense in $V$. □

Theorem I now follows, either by a Mordell-Weil theorem for abelian varieties over function fields, or by bounding the number of holomorphic sections of finite order. Such a bound is given by Corollary 2.7 of [6], which depends on the following lemma (Lemma 2.4 of [6] when the base variety was a quotient of a Hilbert-Siegel space).

LEMMA 5. There is an element $\gamma \in \Gamma$ so that $\det(1 - \gamma) \neq 0$.

We first take a more general setting. Suppose now that $K$ is any number field. Write $K_A$ for the finite adèles of $K$, and write $\mathcal{O}_v$ for the ring of integers of the completion $K_v$. Suppose $H \subset GL_m(C)$ is a linear algebraic group defined over $K$. We have the usual topological groups $H_K, H_A (= H_{K_A}), H_v (= H_{K_v}), H(\mathcal{O}_v) (= H_{\mathcal{O}_v})$. For us, $H$ will be $G_K \cap SL_m(K)$, which satisfies the following two properties:

Strong approximation. $H_K$ is dense in $H_A$.

Local property. For infinitely many primes $w$ of $K$, there is an element $7_w \in H(\mathcal{O}_v)$ so that $\det(1 - 7_w) \neq 0$.

Let $U = \prod_v M_m(\mathcal{O}_v)$, the product over the finite primes of $K$.

THEOREM 6. Suppose $H$ as above is a group with the strong approximation and local properties. Suppose $N \in \mathbb{Z}^+$ and $\Gamma_N = H_K \cap (1 + NU)$. Then there is an element $\gamma \in \Gamma_N$ with $\det(1 - \gamma) \neq 0$.

PROOF. $H_A \cap (1 + NU) = \prod_{v \not| N} H(\mathcal{O}_v) \cdot \prod_{v \mid N} (H_v \cap (1 + NM_n(\mathcal{O}_v)))$. By the local property, we can find a prime $w$ with $w \mid N$, and $\gamma_w \in H(\mathcal{O}_v)$ with $\det(1 - \gamma_w) \neq 0$. Let $U_w = \{\delta \in H(\mathcal{O}_v) | \det(1 - \delta) \neq 0\}$ and let $V = U_w : \prod_{v \not= w} (H_v \cap (1 + NM_n(\mathcal{O}_v)))$, a nonempty open subset of $H_A$. By strong approximation, there is an element $\gamma \in V \cap H_K \subset \Gamma_N$. By the choice of $V$, $\det(1 - \gamma) \neq 0$. □

PROOF OF LEMMA 5. Let $H = G_K \cap SL_m(K)$. Choose positive integers $c, d$, and $R$ so that $d\mathcal{O}_K^m \subset c\mathcal{M} \subset \mathcal{O}_K^m$ and $RN \subset \mathcal{M}$. Then $\Gamma_{Rd} \subset \Gamma$, so it suffices to prove the lemma for $\Gamma_{Rd}$. $H$ has strong approximation by [3 or 1], so by Theorem 6 it suffices to show $H$ has the local property. If $m$ is even, choose any $w$ with $w \mid 2N$, and let $\gamma_w = -1$. If $m$ is odd, choose $w$ so that $\beta \in K$ so that $\alpha^2 \not\in F$. Let $\beta = \alpha / \alpha^2$, and choose $w$ so that $\beta$ is a unit in $\mathcal{O}_w$. Let $\gamma_w = \text{diag}(\beta, \beta, \beta^{-2}, -1, \ldots, -1)$. □

REFERENCES


DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, 231 W. 18 AVENUE, COLUMBUS, OHIO 43210