

STRUCTURE OF GENERALIZED LOCAL RIGID MOTION GROUPS

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ABSTRACT. We consider the higher-order local field analogue of the real euclidean plane afforded by a Galois field extension of degree n , and we describe the structure of its corresponding "rigid motion" group G .

0. Introduction. In 1963 and later in 1967 M. Saito [S1 and S2] constructed the irreducible unitary representations of the groups of "rigid motions" of the p -adic plane determined by a quadratic ramified extension of a locally compact, totally disconnected, nondiscrete field, extending then the work of Vilenkin on the real euclidean plane [V] to the p -adic ramified plane. Inspired by the work of Paul J. Sally, Jr. [Sa], he was able in the second paper to generalize his result to positive characteristic of the base field, but he still had to assume that $p \neq 2$.

It is of interest to extend the group of "rigid motions" of the (local) plane to higher dimensions. One possible generalization is to consider an n -dimensional field extension E of a field F , with norm N , and to look on the function $D: (z, w) \mapsto N(z - w)$ from $E \times E$ to F as an n th order analog of the euclidean metric. We will call the pair (E, D) a "para-euclidean space". A "rigid motion", or isometry, for (E, D) is then any mapping from E to itself which preserves D . We say that an isometry is weakly affine if it sends F into another affine line in E .

In this paper we consider an arbitrary degree Galois extension E of a local field F and we study the structure of the group G of weakly affine isometries of E with respect to the norm of the extension E over F . We prove in fact that the group G is the semidirect product of the translations of E and K , where K is the semidirect product of the group C of multiplications in E by elements of norm 1 and the Galois group Γ , extending thus the well-known structure theorem for the rigid motion group of the real euclidean plane.

The construction of the irreducible smooth representations of our group G is then easily done, with the help of the Mackey Machine as given in [R] (see the note [P-SA], where also a smooth Gel'fand Model for G , in the sense of [G-Z], is constructed for abelian Γ).

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1. The structure theorem for weakly affine isometries of a paraeuclidean space over a local field. Let F be an (archimedean or nonarchimedean) local field and let $E \supset F$ be a Galois extension of degree n . Let $\Gamma = \{\sigma_1 = \text{id}, \sigma_2, \dots, \sigma_n\}$ be the Galois group and $N: E \rightarrow F$ be the norm map of the Galois extension $E \supset F$.

Let us consider the set G of weakly affine isometries of E with respect to N , i.e., the set G of all mappings $\phi: E \rightarrow E$ such that $N(\phi(x) - \phi(y)) = N(x - y)$, for all $x, y \in E$ and $\phi(F) \subset a + bF$ for some $a, b \in E$.

The main purpose of this paper is to prove the following structure theorem:

THEOREM 1. $G = T_E \rtimes (C \rtimes \Gamma)$, where T_E is the group of translations $t_a: x \mapsto x + a$ ($a \in E$) of E and C is the group of multiplications m_u in E by elements u of norm 1.

2. Preliminary lemmas for the proof of Theorem 1. Recall that E^\times has a structure of $\mathbf{Z}[\Gamma]$ -module if we define $w^\eta = \prod_{i=1}^n (\sigma_i(w))^{m_i}$, for w in E^\times and $\eta = \sum_{i=1}^n m_i \sigma_i$ in $\mathbf{Z}[\Gamma]$.

DEFINITION 1. An element z in E^\times is generic if $z^\eta = 1$ implies $\eta = 0$.

PROPOSITION 1. The set of generic elements of E^\times is dense in E^\times .

PROOF. Since local fields are locally compact, by Baire's category theorem it is enough to prove that

$$V_{\vec{m}} = \left\{ z \in E^\times \mid z^{\sum_i m_i \sigma_i} = 1 \right\}$$

has empty interior for each $\vec{m} = (m_1, \dots, m_n)$ in \mathbf{Z}^n . But by picking a basis for E over F , $V_{\vec{m}}$ becomes an (affine) algebraic set in F^n , whence $V_{\vec{m}}$ must have empty interior. \square

LEMMA 1. Let ϕ be an element of G such that $\phi(0) = 0$, $\phi(1) = 1$, and $\phi(F) \subset F$. Then $\phi(t) = t$ for every t in F .

PROOF. Since $N(\phi(x)) = N(x)$ for all x in E and $\phi(F) \subset F$, we have that if s is in F , then $\phi(s) = \varepsilon_s s$, where $\varepsilon_s \in F$ is an n th root of unity.

Now let $s, t \in F$. Then $(\phi(s) - \phi(t))^n = N(\phi(s) - \phi(t)) = N(s - t) = (s - t)^n$, so that $\phi(s) - \phi(t) = \varepsilon_{s,t}(s - t)$ for an n th root of unity $\varepsilon_{s,t}$ in F . We get then $\varepsilon_s s - \varepsilon_t t = \varepsilon_{s,t}(s - t)$, so that $(\varepsilon_s - \varepsilon_{s,t})s = (\varepsilon_t - \varepsilon_{s,t})t$. From this we get that for each s in F^\times , $\varepsilon_t - \varepsilon_{s,t} = 0$ for almost every t in F (for if $\varepsilon_t - \varepsilon_{s,t} \neq 0$, then t belongs to $\{((\varepsilon_s - \varepsilon_{s,t})/(\varepsilon_t - \varepsilon_{s,t}))s \mid \varepsilon_t, \varepsilon_{s,t} \text{ are } n\text{th roots of unity}\}$, which is a finite set). We have then $\varepsilon_s = \varepsilon_{s,t}$ and so $\varepsilon_t = \varepsilon_{s,t} = \varepsilon_s$ for each s and for almost every t (these t 's depending on s). In particular, by taking $s = 1$, we get that $\phi(t) = t$ for almost every t in F . But since ϕ is continuous, we have $\phi(t) = t$ for every t in F . \square

We notice now that if for infinitely many t 's in F we have that $N(z' - t) = N(z - t)$, then z' is conjugate to z , i.e., z' belongs to the set $\{\sigma(z) \mid \sigma \in \Gamma\}$. In particular, for ϕ as in Lemma 1, $\phi(z)$ is conjugate to z .

LEMMA 2. Let ϕ be an element of G such that $\phi(t) = t$ for every t in F and $\phi(z) = z$ for some generic z in E^\times . Then there exists an infinite set $T \subset F$ containing 0 and 1, such that $\phi(tz) = tz$ for all t in T .

PROOF. By the above $\phi(tz) = t\sigma^{(t)}(z)$ for each t in F ; we get in this way a function $t \mapsto \sigma^{(t)}$ from F to Γ . Since F is infinite and Γ is finite, there exist infinitely many t 's in F mapped to a single σ in Γ . For these t 's and this σ we have

$$\begin{aligned} N\left(\frac{z}{\sigma(z)} - t\right) &= N\left(\frac{\sigma(z)}{z}\right)N\left(\frac{z}{\sigma(z)} - t\right) = N\left(1 - t\frac{\sigma(z)}{z}\right) \\ &= \frac{N(z - t\sigma(z))}{N(z)} = \frac{N(\phi(z) - \phi(tz))}{N(z)} = \frac{N(z - tz)}{N(z)} = N(1 - t); \end{aligned}$$

thus $z/\sigma(z) = 1$ and so $\phi(tz) = t\sigma(z) = tz$. \square

LEMMA 3. *If ϕ is as in Lemma 2, for all σ in Γ we have $\phi(\sigma(z)) = \sigma(z)$.*

PROOF. Let σ_i be an element of Γ ; then if t is in T

$$\begin{aligned} N\left(\frac{\phi(\sigma_i(z))}{z} - t\right) &= N\left(\frac{\phi(\sigma_i(z)) - tz}{z}\right) = \frac{N(\phi(\sigma_i(z)) - \phi(tz))}{N(z)} \\ &= \frac{N(\sigma_i(z) - tz)}{N(z)} = N\left(\frac{\sigma_i(z)}{z} - t\right), \end{aligned}$$

and so $\phi(\sigma_i(z))/z = \sigma_j(\sigma_i(z))/\sigma_j(z)$ for some σ_j in Γ ; on the other hand $\phi(\sigma_i(z)) = \sigma_k(\sigma_i(z))$ for certain σ_k in Γ . Thus

$$\frac{\sigma_k(\sigma_i(z))}{z} = \frac{\sigma_j(\sigma_i(z))}{\sigma_j(z)}.$$

Since z is generic we must have $\sigma_j = \sigma_k = \text{id}$ and so $\phi(\sigma_i(z)) = \sigma_i(z)$. \square

3. Proof of Theorem 1. Let ϕ be an element of G . The condition $\phi(F) \subset a + bF$ allows us, after translation and scaling, to assume that $\phi(0) = 0$, $\phi(1) = 1$, and $\phi(F) \subset F$. We can then apply Lemma 1 and get that $\phi(t) = t$ for every t in F . Now by Proposition 1, we may take a generic element z of E and we can use Lemma 2 to get that $\phi(tz) = tz$ for all t in T . Similarly, we can get that there exist infinitely many t 's in F such that $\phi(t\sigma(z)) = t\sigma(z)$.

In order to prove our theorem we need to prove that $\phi = \text{id}$. To this end let $w \in E$ and let $w' = \phi(w)$. We want to prove then that $w' = w$.

From

$$N\left(\frac{w'}{\sigma_i(z)} - t\right) = N\left(\frac{w}{\sigma_i(z)} - t\right) \quad \text{for } i = 1, \dots, n,$$

we get that

$$\frac{w'}{\sigma_i(z)} \in \left\{ \frac{w}{\sigma_i(z)}, \frac{\sigma_2(w)}{\sigma_2(\sigma_i(z))}, \dots, \frac{\sigma_n(w)}{\sigma_n(\sigma_i(z))} \right\} \quad \text{for each } i = 1, \dots, n.$$

In other words for each i , $1 \leq i \leq n$, there is an index j_i such that $w'/\sigma_{j_i}(w) = \sigma_i(z)/\sigma_{j_i}(\sigma_i(z))$.

If $w' \neq w$ then $j_i \neq 1$ for each i , $1 \leq i \leq n$. Then necessarily we have two indices i_1 and i_2 , $i_1 \neq i_2$, such that $j_{i_1} = j_{i_2} = j$, and so $\sigma_{i_1}(z)/\sigma_j(\sigma_{i_1}(z)) = \sigma_{i_2}(z)/\sigma_j(\sigma_{i_2}(z))$, which contradicts the genericity of z . So we must have $w' = w$ and $\phi = \text{id}$.

Summing up, our argument shows that if $\phi \in G$ then $\text{id} = \sigma^{-1}m_{u-1}t_{-a}\phi$ for certain u of norm 1 and some $a \in E$; i.e., $\phi = t_a m_u \sigma$. In other words $G = T_E \cdot C \cdot \Gamma$ and G is a group. Moreover, it is then clear that $G = T_E \rtimes (C \rtimes \Gamma)$. \square

4. Remarks.

4.1. *The finite field case.* Theorem 1 does not hold for finite field extensions of degree bigger than 2. For instance, for $F = \mathbf{F}_2$ and $E = \mathbf{F}_{2^n}$, we have that G is simply the group of all bijections of E , which coincides with its subgroup $T_E \rtimes (C \rtimes \Gamma)$ only for $n \leq 2$.

4.2. *The general quadratic case.* On the other hand, we remark that Theorem 1 holds for any quadratic extension E of an arbitrary base field F , finite or infinite.

To see this, we first reduce, by translation and scaling, as above, to the case in which our isometry ϕ fixes 0 and 1. But then, for any $x \in E$, we have

$$N(\phi(x) - 1) = N(x - 1), \quad N(\phi(x)) = N(x).$$

It follows that also $\text{Tr}(\phi(x)) = \text{Tr}(x)$, whence $\phi(x) = \sigma_x(x)$ for some $\sigma_x \in \Gamma$ depending on x .

Fix $z \in E$ which is not in F . By composing with σ_z^{-1} , we reduce to the case in which ϕ fixes 0, 1 and z . We want to prove then that $\phi = \text{id}$. Pick $w \in E$ and suppose that $\phi(w) \neq w$. Then $\sigma := \sigma_w \neq \text{id}$ and so both $\Gamma = \{\text{id}, \sigma\}$ and the trace map Tr from E to F are nontrivial. But we must have $\text{Tr}(\phi(w) - w) = 0$, and also $\text{Tr}((\phi(w) - w)\sigma(z)) = 0$ since $N(\phi(w) - z) = N(w - z)$. It follows that the F -linear map $x \mapsto \text{Tr}((\phi(w) - w)\sigma(x))$ from E to F is identically zero, whence $\phi(w) = w$. So we have proved

THEOREM 2. *For any quadratic extension E of an arbitrary base field F , the corresponding isometry group G for the "metric" $D: (z, w) \mapsto N(z - w)$ on E is the semidirect product $T_E \rtimes (C \rtimes \Gamma)$. \square*

Notice that we do not need to assume here that our isometries ϕ are weakly affine. Moreover, in the nonseparable case, in which $\Gamma = \{\text{id}\}$, Theorem 2 boils down to the obvious fact that $G = T_E$.

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