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# INVARIANT SUBSPACES FOR ALGEBRAS OF SUBNORMAL OPERATORS. II

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ABSTRACT. We continue our study of hyperinvariant subspaces for rationally cyclic subnormal operators. We establish a connection between hyperinvariant subspaces and weak-star continuous point evaluations on the commutant.

**Introduction.** Let $K$ be a compact subset of the complex plane $\mathbf{C}$ and let $R(K)$ denote the algebra of rational functions with poles off $K$. For a positive measure $\mu$ with support in $K$ let $R^2(K,\mu)$ denote the closure of $R(K)$ in $L^2(\mu)$. Every rationally cyclic subnormal operator is unitarily equivalent to multiplication by $z$, $M_z$, on an $R^2(K,\mu)$-space [**1**, p. 146]. Under this representation each operator that commutes with $M_z$ is represented by multiplication by a function in $R^2(K,\mu) \cap L^\infty(\mu)$, and conversely [**1**, p. 147].

In [**2**] we proved the existence of invariant subspaces for the algebra $R^2(K,\mu) \cap L^\infty(\mu)$. In this paper we show there exist weak-star continuous point evaluations on $R^2(K,\mu) \cap L^\infty(\mu)$ if $L^2(\mu) \neq R^2(K,\mu)$, and we show that these point evaluations give rise to the kinds of invariant subspaces found in [**2**]. Finally, we establish a connection between analytic bounded point evaluations on $R^2$-spaces and $R^p$-spaces for $2 < p < 4$.

*Notation.* Let $m$ denote area measure on $\mathbf{C}$. For a complex measure $\nu$ with compact support in $\mathbf{C}$ let $\hat{\nu}(z) = \int (\varsigma - z)^{-1}\, d\nu(\varsigma)$ and $\tilde{\nu}(z) = \int |\varsigma - z|^{-1}\, d|\nu|(\varsigma)$. By Tonelli's theorem $\tilde{\nu}(z) < \infty$ $m$-a.e. and hence the Cauchy transform $\hat{\nu}$ is defined $m$-a.e. For $g$ in $L^1(|\nu|)$ let $\hat{g}$ equal $(g\, d\nu)\hat{}$.

THEOREM. *Let $\overline{g} \in R^2(K,\mu)^\perp$ and let $A = \{z\colon \hat{g}(z) \neq 0\}$. Then for $m$-a.e. $z$ in $A$ there exists a weak-star continuous multiplicative linear functional $e_z$ on $R^2(K,\mu) \cap L^\infty(\mu)$ such that $e_z(f) = f(z)$ for each $f$ in $R(K)$. Moreover, for $m$-a.e. such $z$ there exist $x$ and $y$ in $R^2(K,\mu)$ such that $e_z(f) = \langle fx, y\rangle$ for each $f$ in $R^2(K,\mu) \cap L^\infty(\mu)$.*

REMARK. Let $z$ in $A$ be such that both conclusions of the theorem hold, and let $x$ and $y$ be as in the second conclusion. Let $H$ be the closed linear span of $\{(\varsigma - z)fx\colon f \in R^2(K,\mu) \cap L^\infty(\mu)\}$ in $L^2(\mu)$. Since $y \perp H$, it follows that $H$ is a nontrivial hyperinvariant subspace for $M_\varsigma$ on $R^2(K,\mu)$.

LEMMA 1. *Let $p \in (2,4)$ and let $s = 2/(p-2)$. Suppose $f_n \in L^2(\mu)$ and $\|f_n\|_2 < 2^{-n}$ for each positive integer $n$. Then there exists a function $w\colon \mathbf{C} \to (0,1]$ such that $w^{-1} \in L^s(\mu)$, $f_n \in L^p(w\, d\mu)$ for each $n$, and $f_n \to 0$ in $L^p(w\, d\mu)$.*

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PROOF. Let $w = (1 + \sum_{n=1}^{\infty} |f_n|)^{2-p}$. Redefine $w$ to be one on the set where the infinite series diverges. Since $w^{-s} = (1 + \sum |f_n|)^2$, it follows that $w^{-s} \in L^1(\mu)$ and hence $w^{-1} \in L^s(\mu)$. Finally, $|f_n|^p w \leq |f_n|^2$ implies that $f_n \in L^p(w\,d\mu)$ for each $n$ and that $f_n \to 0$ in $L^p(w\,d\mu)$.

LEMMA 2. *Let $p \in (2,4)$, $s = 2/(p-2)$, and $q = p(p-1)^{-1}$. Suppose $R^2(K,\mu) \neq L^2(\mu)$. Let $f_1, f_2 \in R^2(K,\mu)$. Then there exists a function $w\colon \mathbf{C} \to (0,1]$ such that $w^{-1} \in L^s(\mu)$, $R^p(K, w\,d\mu) \neq L^p(w\,d\mu)$, and $f_1, f_2 \in R^p(K, w\,d\mu)$.*

PROOF. Define $f_n$ to be $f_1$ when $n$ is odd and $f_2$ when $n$ is even. Let $\{r_n\}$ be a sequence of functions from $R(K)$ such that $\|r_n - f_n\|_2 < 2^{-n}$ for each $n$. Applying Lemma 1 to the sequence $\{r_n - f_n\}$, we obtain a function $w\colon \mathbf{C} \to (0,1]$ such that $w^{-1} \in L^s(\mu)$ and $r_n - f_n \to 0$ in $L^p(w\,d\mu)$. It follows immediately that $f_1$ and $f_2$ are in $R^p(K, w\,d\mu)$. If $\overline{g} \in R^2(K,\mu)^\perp$ then $(g/w) \in L^q(w\,d\mu)$ by Hölder's inequality. Since $R^2(K,\mu) \neq L^2(\mu)$, it follows that $R^p(K, w\,d\mu) \neq L^p(w\,d\mu)$.

PROOF OF THE THEOREM. Fix $p$ and $q$ as in Lemma 2. Let $B$ be a compact subset of $A$ such that the following hold:

(i) $(g\,d\mu)^{\sim}|_B$ is continuous;

(ii) $(\varsigma - z)^{-1} g(\varsigma) \in L^q(\mu)$ for each $z \in B$.

As a consequence of (i), $\hat{g}|_B$ is continuous and hence $\hat{g}$ is bounded away from zero on $B$. Let $E$ be the set of points in $B$ that are of full density with respect to $m$. We shall show that both conclusions of the theorem hold for each $z$ in $E$. We leave it as an exercise that $m$-a.e. $z$ in $A$ belongs to such an $E$.

For $z \in E$ let $e_z(f) = \hat{g}(z)^{-1} \int (\varsigma - z)^{-1} f(\varsigma) g(\varsigma)\, d\mu(\varsigma)$ for each $f \in R^2(K,\mu) \cap L^\infty(\mu)$. It is easy to check that $e_z(f) = f(z)$ if $f \in R(K)$. By the main proof in [**2**] there exist $x \in R^2(K,\mu)$ and $y \in L^2(\mu)$ such that $e_z(f) = \langle fx, y\rangle$ for each $f$ in $R^2(K,\mu) \cap L^\infty(\mu)$. Projecting $y$ into $R^2(K,\mu)$, we may assume that $y \in R^2(K,\mu)$. It remains to show that $e_z$ is multiplicative on $R^2(K,\mu) \cap L^\infty(\mu)$.

For $f \in R^2(K,\mu) \cap L^\infty(\mu)$ the map $z \to e_z(f)$ is continuous on $E$. This is a consequence of assumption (i). Thus it suffices to show that for $f_1, f_2 \in R^2(K,\mu) \cap L^\infty(\mu)$ there exists a dense subset $F$ of $E$ such that $e_z(f_1 f_2) = e_z(f_1) e_z(f_2)$ for each $z \in F$. Let $F$ be the set of $z \in E$ for which

$$\int |\varsigma - z|^{-q} |(g/w)|^q w\, d\mu < \infty.$$

By Tonelli's theorem $m(E \backslash F) = 0$. For $z \in F$ the map $f \to e_z(f)$ is $L^p(w\,d\mu)$-continuous on $R(K)$. Since $e_z$ is multiplicative on $R(K)$, it follows from taking $L^p(w\,d\mu)$ limits that $e_z(f_1 f) = e_z(f_1) e_z(f)$ for each $f \in R(K)$. Taking $L^p(w\,d\mu)$ limits again, we obtain $e_z(f_1 f_2) = e_z(f_1) e_z(f_2)$. That does it.

REMARK. Let $P^r(\mu)$ denote the closure of the polynomials in $L^r(\mu)$. Suppose $P^2(\mu)$ has no analytic bounded point evaluations. Then there exists a function $f$ in $P^2(\mu)$ that is essentially unbounded on every relatively open subset of the support of $\mu$. If $P^2(\mu) \neq L^2(\mu)$ then by Lemma 2 $f$ is in a nontrivial $P^r$-space for every $r \in (2,4)$. It follows that none of those $P^r$-spaces has any analytic bounded point evaluations. Thus, to show that every nontrivial $P^2$-space has some analytic bounded point evaluations, it suffices to show that every nontrivial $P^r$-space has analytic bounded point evaluations for some $r \in (2,4)$. It is true that bounded point evaluations exist for every nontrivial $P^r$-space for $r > 2$.

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