INVARIANT SUBSPACES FOR ALGEBRAS
OF SUBNORMAL OPERATORS. II

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ABSTRACT. We continue our study of hyperinvariant subspaces for rationally
cyclic subnormal operators. We establish a connection between hyperinvariant
subspaces and weak-star continuous point evaluations on the commutant.

Introduction. Let \( K \) be a compact subset of the complex plane \( \mathbb{C} \) and let \( R(K) \)
denote the algebra of rational functions with poles off \( K \). For a positive measure
\( \mu \) with support in \( K \) let \( R^2(K, \mu) \) denote the closure of \( R(K) \) in \( L^2(\mu) \). Every
rationally cyclic subnormal operator is unitarily equivalent to multiplication by \( z \),
\( M_z \), on an \( R^2(K, \mu) \)-space [1, p. 146]. Under this representation each operator that
commutes with \( M_z \) is represented by multiplication by a function in \( R^2(K, \mu) \cap
L^\infty(\mu) \), and conversely [1, p. 147].

In [2] we proved the existence of invariant subspaces for the algebra \( R^2(K, \mu) \cap
L^\infty(\mu) \). In this paper we show there exist weak-star continuous point evaluations on
\( R^2(K, \mu) \cap L^\infty(\mu) \) if \( L^2(\mu) \neq R^2(K, \mu) \), and we show that these point evaluations
give rise to the kinds of invariant subspaces found in [2]. Finally, we establish a
connection between analytic bounded point evaluations on \( R^2 \)-spaces and \( R^p \)-spaces
for \( 2 < p < 4 \).

Notation. Let \( m \) denote area measure on \( \mathbb{C} \). For a complex measure \( \nu \) with
compact support in \( \mathbb{C} \) let \( \tilde{\nu}(z) = \int (\xi - z)^{-1} d\nu(\xi) \) and \( \tilde{\nu}(z) = \int |\xi - z|^{-1} d|\nu|(\xi) \).
By Tonelli’s theorem \( \tilde{\nu}(z) < \infty \) \( m \)-a.e. and hence the Cauchy transform \( \tilde{\nu} \) is defined
\( m \)-a.e. For \( g \) in \( L^1(|\nu|) \) let \( \hat{g} \) equal \( (g d\nu)^{-} \).

THEOREM. Let \( g \in R^2(K, \mu) \) and let \( A = \{z: g(z) \neq 0\} \). Then for \( m \)-a.e.
z in \( A \) there exists a weak-star continuous multiplicative linear functional \( e_z \) on
\( R^2(K, \mu) \cap L^\infty(\mu) \) such that \( e_z(f) = f(z) \) for each \( f \) in \( R(K) \). Moreover, for \( m-\)
a.e. such \( z \) there exist \( x \) and \( y \) in \( R^2(K, \mu) \) such that \( e_z(f) = (fx, y) \) for each \( f \) in
\( R^2(K, \mu) \cap L^\infty(\mu) \).

REMARK. Let \( z \) in \( A \) be such that both conclusions of the theorem hold, and
let \( x \) and \( y \) be as in the second conclusion. Let \( H \) be the closed linear span of
\( \{(\xi - z)f x: f \in R^2(K, \mu) \cap L^\infty(\mu)\} \) in \( L^2(\mu) \). Since \( y \perp H \), it follows that \( H \) is a
nontrivial hyperinvariant subspace for \( M_z \) on \( R^2(K, \mu) \).

LEMMA 1. Let \( p \in (2,4) \) and let \( s = 2/(p - 2) \). Suppose \( f_n \in L^2(\mu) \) and
\( \|f_n\|_2 < 2^{-n} \) for each positive integer \( n \). Then there exists a function \( w: \mathbb{C} \to (0,1] \)
such that \( w^{-1} \in L^s(\mu) \), \( f_n \in L^p(\mu d\mu) \) for each \( n \), and \( f_n \to 0 \) in \( L^p(\mu d\mu) \).
PROOF. Let \( w = (1 + \sum_{n=1}^{\infty} |f_n|)^{2-p} \). Redefine \( w \) to be one on the set where the infinite series diverges. Since \( w^{-s} = (1 + \sum |f_n|)^2 \), it follows that \( w^{-s} \in L^1(\mu) \) and hence \( w^{-1} \in L^s(\mu) \). Finally, \( |f_n|^p w \leq |f_n|^2 \) implies that \( f_n \in L^p(w \, d\mu) \) for each \( n \) and that \( f_n \rightarrow 0 \) in \( L^p(w \, d\mu) \).

LEMMA 2. Let \( p \in (2,4) \), \( s = 2/(p-2) \), and \( q = p(p-1)^{-1} \). Suppose \( R^2(K,\mu) \neq L^2(\mu) \). Let \( f_1, f_2 \in R^2(K,\mu) \). Then there exists a function \( w : C \rightarrow (0,1] \) such that \( w^{-1} \in L^s(\mu) \), \( R^p(K, w \, d\mu) \neq L^p(w \, d\mu) \), and \( f_1, f_2 \in R^p(K, w \, d\mu) \).

PROOF. Define \( f_n \) to be \( f_1 \) when \( n \) is odd and \( f_2 \) when \( n \) is even. Let \( \{r_n\} \) be a sequence of functions from \( R(K) \) such that \( ||r_n - f_n||_2 < 2^{-n} \) for each \( n \). Applying Lemma 1 to the sequence \( \{r_n - f_n\} \), we obtain a function \( w : C \rightarrow (0,1] \) such that \( w^{-1} \in L^s(\mu) \) and \( r_n - f_n \rightarrow 0 \) in \( L^p(w \, d\mu) \). It follows immediately that \( f_1 \) and \( f_2 \) are in \( R^p(K, w \, d\mu) \). If \( g \in R^2(K,\mu)^{-1} \) then \( (g/w) \in L^q(w \, d\mu) \) by Hölder’s inequality. Since \( R^2(K,\mu) \neq L^2(\mu) \), it follows that \( R^p(K, w \, d\mu) \neq L^p(w \, d\mu) \).

PROOF OF THE THEOREM. Fix \( p \) and \( q \) as in Lemma 2. Let \( B \) be a compact subset of \( A \) such that the following hold:

(i) \( (g \, d\mu)^{-1}|_B \) is continuous;
(ii) \( (\zeta - z)^{-1}g(\zeta) \in L^q(\mu) \) for each \( z \in B \).

As a consequence of (i), \( \hat{g}|_B \) is continuous and hence \( \hat{g} \) is bounded away from zero on \( B \). Let \( E \) be the set of points in \( B \) that are of full density with respect to \( m \).

We shall show that both conclusions of the theorem hold for each \( z \in E \). We leave it as an exercise that \( m \)-a.e. \( z \) in \( A \) belongs to such an \( E \).

For \( z \in E \) let \( e_z(f) = \hat{g}(z)^{-1} \int (\zeta - z)^{-1}f(\zeta)g(\zeta) \, d\mu(\zeta) \) for each \( f \in R^2(K,\mu) \cap L^\infty(\mu) \). It is easy to check that \( e_z(f) = f(z) \) if \( f \in R(K) \). By the main proof in [2] there exist \( x \in R^2(K,\mu) \) and \( y \in L^2(\mu) \) such that \( e_z(f) = (fx,y) \) for each \( f \) in \( R^2(K,\mu) \cap L^\infty(\mu) \). Projecting \( y \) into \( R^2(K,\mu) \), we may assume that \( y \in R^2(K,\mu) \).

It remains to show that \( e_z \) is multiplicative on \( R^2(K,\mu) \cap L^\infty(\mu) \).

For \( f \in R^2(K,\mu) \cap L^\infty(\mu) \) the map \( z \rightarrow e_z(f) \) is continuous on \( E \). This is a consequence of assumption (i). Thus it suffices to show that for \( f_1, f_2 \in R^2(K,\mu) \cap L^\infty(\mu) \) there exists a dense subset \( F \) of \( E \) such that \( e_z(f_1f_2) = e_z(f_1)e_z(f_2) \) for each \( z \in F \). Let \( F \) be the set of \( z \in E \) for which

\[
\int |\zeta - z|^{-q} |(g/w)|^q w \, d\mu < \infty.
\]

By Tonelli’s theorem \( m(E \setminus F) = 0 \). For \( z \in F \) the map \( f \rightarrow e_z(f) \) is \( L^p(w \, d\mu) \)-continuous on \( R(K) \). Since \( e_z \) is multiplicative on \( R(K) \), it follows from taking \( L^p(w \, d\mu) \) limits that \( e_z(f_1f_2) = e_z(f_1)e_z(f_2) \) for each \( f \in R(K) \). Taking \( L^p(w \, d\mu) \) limits again, we obtain \( e_z(f_1f_2) = e_z(f_1)e_z(f_2) \). That does it.

REMARK. Let \( P^r(\mu) \) denote the closure of the polynomials in \( L^r(\mu) \). Suppose \( P^2(\mu) \) has no analytic bounded point evaluations. Then there exists a function \( f \) in \( P^2(\mu) \) that is essentially unbounded on every relatively open subset of the support of \( \mu \). If \( P^2(\mu) \neq L^2(\mu) \) then by Lemma 2 \( f \) is in a nontrivial \( P^r \)-space for every \( r \in (2,4) \). It follows that none of those \( P^r \)-spaces has any analytic bounded point evaluations. Thus, to show that every nontrivial \( P^2 \)-space has some analytic bounded point evaluations, it suffices to show that every nontrivial \( P^r \)-space has analytic bounded point evaluations for some \( r \in (2,4) \). It is true that bounded point evaluations exist for every nontrivial \( P^r \)-space for \( r > 2 \).
REFERENCES


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