NONEXPANSIVE ACTIONS OF TOPOLOGICAL SEMIGROUPS
ON STRICTLY CONVEX BANACH SPACES
AND FIXED POINTS

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ABSTRACT. Let C be a closed convex subset of a strictly convex Banach space X and \( \{T_s : s \in S\} \) be a continuous representation of a semitopological semigroup S as nonexpansive mappings of C into itself. The main result establishes the fact that if for some \( x \in C \) the trajectory \( \{T_s x : s \in S\} \) is relatively compact and \( AP(S) \) has a left invariant mean then \( K = \text{conv}\{T_s x : s \in S\} \) contains a common fixed point for \( \{T_s\}_s \).

Let C be a closed, convex subset of a Banach space X, and T be a nonexpansive mapping of C into C (i.e., \( \|Tx - Ty\| \leq \|x - y\| \) for every \( x, y \in C \)). For \( x \in C \) the orbit of x is the set \( \mathcal{O}(x) = \{T^n x : n \geq 0\} \) and the \( \omega \)-limit set of x is defined by \( \omega(x) = \{y : \lim T^{n_k} x = y \text{ for some subsequence } n_k \} \). The \( \omega \)-limit set of a point x is easily shown to be closed and \( T \)-invariant although possibly empty. If \( \omega(x) \) is nonempty then it is minimal (the orbit \( \mathcal{O}(y) \) is a dense subset of \( \omega(x) \) for every \( y \in \omega(x) \)) and the action of T on \( \omega(x) \) is isometric (see [D.S.]). By [R.S.], nonempty \( \omega(x) \) can be given the structure of a monothetic group. So there exists a \( T \)-invariant probability measure \( \mu \) (\( \mu \) is invariant if \( \mu \circ T^{-1} = \mu \)) on \( \omega(x) \) if and only if \( \omega(x) \) is (nonempty) compact (see also [B.D.]). The following lemma gives connections between the existence of compact orbits and \( T \)-invariant measures in slightly more general situations.

**Lemma 1.** Let \( X, C, T \) be as above. If C is separable and \( \mu \) is a \( T \)-invariant probability (on a Borel \( \sigma \)-field) then every \( x \in \text{supp} \mu \) is recurrent and \( \omega(x) \) is compact (\( \text{supp} \mu \) denotes here the smallest closed subset of C of full measure \( \mu \)).

**Proof.** Notice that \( \text{supp} \mu \) is a \( T \)-invariant subset of C (i.e., if \( x \in \text{supp} \mu \) then \( Tx \in \text{supp} \mu \)). By the classical Poincaré recurrence theorem, the set of recurrent points is full measure, so the set of recurrent points is dense in \( \text{supp} \mu \). Since T is nonexpansive, every \( x \in \text{supp} \mu \) is recurrent. Thus \( \text{supp} \mu = \bigcup_{x \in \text{supp} \mu} \omega(x) \), and for every \( x, y \in \text{supp} \mu \) the limit sets \( \omega(x), \omega(y) \) coincide or are disjoint. In order to show compactness of \( \omega(x) \) it is enough to show the existence of \( T \)-invariant probability on \( \omega(x) \). Let \( \mathcal{F} \) denote the partition of \( W = \text{supp} \mu \) on the sets \( \omega(x) \). It is known (see [R] or [P]) that there exists a system of canonical measures \( \mu_\tau \) concentrated on \( \tau (\tau = \omega(x) \text{ for } x \in W) \) such that

\[
\mu = \int_{W/\mathcal{F}} \mu_\tau \nu(d\tau)
\]
and $\nu$ is some probability measure on $W/\mathcal{F}$. Let $\sigma_{\mathcal{F}}$ denote the sub $\sigma$-field generated by sets of the partition $\mathcal{F}$. Clearly

$$\int f \, d\mu(x) = E(f | \sigma_{\mathcal{F}})(x) = E(f | \sigma_{\mathcal{F}})(T(x))$$

$$= E(f \circ T | \sigma_{\mathcal{F}})(x) = \int f \circ T \, d\mu(x),$$

for $\nu$ almost all $\omega(x) \in W/\mathcal{F}$ ($f$ is here an arbitrary continuous function on $\text{supp} \mu$). Thus, for $\nu$ and almost all $\tau$, the measures $\mu_{\tau}$ are $T$-invariant, so $\omega(x)$ is compact for $x$ from a dense subset of $W$ (see [R.S.] or [B.D.]). But the set $\{ y : \omega(y) \text{ is compact} \}$ is closed (and convex if $X$ is strictly convex), and thus for all $x \in \text{supp} \mu$ the orbit $\omega(x) = \overline{\mathcal{O}(x)}$ is compact.

**PROPOSITION.** Let $X$ be a strictly convex Banach space and $C$ be a separable, convex, closed subset of $X$. If $T : C \to C$ is nonexpansive then the following conditions are equivalent:

(i) the set $F(T)$ of fixed points of $T$ is nonempty;

(ii) there exists a $T$-invariant probability;

(iii) there exists $x \in C$ such that $\overline{\mathcal{O}(x)}$ is compact.

**Proof.** Because (i)$\Rightarrow$(ii) and (iii)$\Rightarrow$(ii) are trivial and (ii)$\Rightarrow$(iii) follows from our lemma we only have to show (ii)$\Rightarrow$(i). Let $\mu$ be the $T$-invariant probability measure on $\omega(x)$ (unique by nonexpansiveness of $T$). Since $T$ is affine on $\text{conv} \, \omega(x)$ (see [E] or [Y]), for every $x^* \in X^*$, $x \circ T$ is affine, continuous and $x^* \circ (\text{bar} \, \mu) = \int_{\omega(x)} x^*(y) \mu(dy) = \int_{\omega(x)} x^* \circ T(y) \mu(dy) = x^* \circ T(\text{bar} \, \mu) = x^* (T(\text{bar} \, \mu))$. Since the functionals separate points of $X$ the barycenter of $\mu$ is a fixed point of $T$.

Let $S$ be a semitopological semigroup, i.e., $S$ is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \to as$ and $s \to sa$ from $S$ to $S$ are continuous. Let $X$ be a strictly convex Banach space, and $S \ni s \to T_s$ be a continuous representation of $S$ as nonexpansive mappings on a closed convex subset $C$ of $X$ into $C$, i.e., $T_{ab}(x) = T_a(T_b(x))$, $a,b \in S$, $x \in C$, and the mapping $s \to T_s x$ from $S$ into $C$ is continuous for every $x \in C$. If $f \in C(S)$ and $a \in S$ define $l_a f(s) = f(as)$ and $r_a f(s)$ for all $s \in S$ ($C(S)$ denotes here the set of all continuous bounded functions on $S$). Recall, a function $f \in C(S)$ is said to be almost periodic on $S$ if $\{ r_a f : a \in S \}$ is relatively compact in the norm topology of $C(S)$. The subalgebra of all almost periodic functions on $S$ we denote by $AP(S)$.

A linear functional $m \in AP(S)^*$ is called a left invariant mean if for all $a \in S$ and $f \in AP(S)$ we have $\langle l_a f, m \rangle = \langle f, m \rangle$ and $\langle 1, m \rangle = 1$. The following theorem is a partial solution of Problem 2 from [L.2].

**THEOREM.** Let $\{ T_s : s \in S \}$ be a continuous representation of a semitopological semigroup $S$ as nonexpansive mappings on a closed convex subset of a strictly convex Banach space $X$. If $AP(S)$ has a left invariant mean, $x \in C$ such that $\{ T_s x : s \in S \}$ is relatively compact then $K = \text{conv} \{ T_s x : s \in S \}$ contains a common fixed point for $\{ T_s \}_{s \in S}$.

**Proof.** It is clear that for every continuous function $f$ on $C$ the function $\tilde{f}$ defined on $S$ as $\tilde{f}(s) = f(T_s x)$ belongs to $AP(S)$. Thus a left invariant mean $m$ on $AP(S)$ defines a probability measure $\mu$ on $\{ T_s x : s \in S \}$. Clearly (see [L.1]), the measure $\mu$
is $T_s$-invariant for every $s \in S$. By Lemma 1, if $y \in \text{supp} \mu$ then for every $s \in S$, $y$ is $T_s$ recurrent and $T_s$ is affine on $K' = \text{conv}(\text{supp} \mu) \subseteq K$. But by our proposition the barycenter of measure $\text{bar}(\mu) \in K'$ is a fixed point for every $T_s$.

It is a pleasure to thank Professors Lau and Sine for sending me preprints of some of their works. These lead to the following remark: Problem 2 of Lau [L.2] has a negative answer in general. There is an appropriate nonexpansive map $T$ in a 3-dimensional (Banach) space for which $(N+1)^{-1}(I+T+\cdots+T^N)x$ converges, but the limit is not a fixed point, and there is no fixed point in the closed convex hull of the orbit (see Robert Sine, *Behaviour of iterates in the Poincare metric*, preprint, 1986).

**REFERENCES**


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