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MOST QUASIDIAGONAL OPERATORS ARE NOT BLOCK-DIAGONAL

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ABSTRACT. The set of all block-diagonal operators is a dense first category subset of the class (QD) of all quasidiagonal operators. On the other hand, the subset of all irreducible quasidiagonal operators with thin spectra, that are similar to block-diagonal ones, includes a G_δ-dense subset of (QD).

C.-K. Fong proved that most normal operators are diagonal, in the sense that the class of all normal operators includes a G_δ-dense subset of diagonal ones [**1**]. By using his results in [**6**], the author proved that Fong's argument can be modified to show that most quasitriangular operators are triangular and most biquasitriangular operators are bitriangular ("most" in the same sense as above [**7**]). Here *operator* means a bounded linear mapping from a complex, separable, infinite dimensional Hilbert space $\mathscr{H}$ into itself; $T \in \mathscr{L}(\mathscr{H})$ (:= the algebra of all operators acting on $\mathscr{H}$) is called *triangular* if it admits a representation as an upper triangular matrix with respect to a suitable orthonormal basis of $\mathscr{H}$. T is *bitriangular* if both T and its adjoint T^* are triangular (not necessarily with respect to the same basis, of course).

Recall that $T \in \mathscr{L}(\mathscr{H})$ is *quasidiagonal* if there exists an increasing sequence $\{P_n\}_{n=1}^\infty$ of finite rank orthogonal projections such that

$$P_n \to 1 \quad (\textit{strongly}) \quad \text{and} \quad \|TP_n - P_nT\| \to 0 \quad (n \to \infty).$$

An operator B is *block-diagonal* if there exists $\{P_n\}_{n=1}^\infty$ as above such that $BP_n = P_nB$ for all $n = 1, 2, \dots$ (and therefore $B = \bigoplus_{n=1}^\infty B_n$, where $B_n = (P_n - P_{n-1})B \mid \operatorname{ran}(P_n - P_{n-1})$, $n = 1, 2, \dots$; $P_0 = 0$).

The results of [**1** and **7**] might suggest that most quasidiagonal operators (class (QD)) are block-diagonals (class (BD)). But this is definitely false. *Reason.* If $\mathscr{K}(\mathscr{H})$ denotes the ideal of compact operators, then $(QD) + \mathscr{K}(\mathscr{H}) = (BD) + \mathscr{K}(\mathscr{H}) = (BD)^- = (QD)$ [**3**]. (As usual, the upper bar denotes norm-closure.) P. R. Halmos has shown that the *irreducible operators* form a G_δ-dense subset of $\mathscr{L}(\mathscr{H})$ [**2**] (see also [**10**]; $A \in \mathscr{L}(\mathscr{H})$ is irreducible if there is no nontrivial subspace $\mathscr{M}$ of $\mathscr{H}$ such that $A\mathscr{M} \subset \mathscr{M}$ and $A(\mathscr{H} \ominus \mathscr{M}) \subset \mathscr{H} \ominus \mathscr{M}$). Furthermore, Halmos actually proved that given T in $\mathscr{L}(\mathscr{H})$ and $\varepsilon > 0$, there exists $K_\varepsilon \in \mathscr{K}(\mathscr{H})$, with $\|K_\varepsilon\| < \varepsilon$, such that $T - K_\varepsilon$ is irreducible. Since (QD) is a closed subset of $\mathscr{L}(\mathscr{H})$, invariant under compact perturbations, it is not difficult to check that Halmos's argument also proves that the irreducible quasidiagonal operators form

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a G_δ-dense subset of (QD). Since the block-diagonal operators are reducible, we conclude that (BD) is a *first category* subset of (QD).

(Incidentally: the class of all normal operators does not have this kind of problem because it is a closed subset of $\mathcal{L}(\mathcal{H})$, disjoint from the class of all irreducible operators, and not invariant under compact perturbations.)

Let $\sigma(T)$ denote the spectrum of $T \in \mathcal{L}(\mathcal{H})$, and let $\sigma_e(T)$ denote its essential spectrum, that is, the spectrum of $T + \mathcal{K}(\mathcal{H})$ in the quotient Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Recall that $\lambda \in \sigma(T)$ is a *normal eigenvalue* if λ is an isolated point of $\sigma(T)$ and the Riesz spectral invariant subspace $\mathcal{H}(T;\lambda)$ corresponding to the clopen subset $\{\lambda\}$ of $\sigma(T)$ is finite dimensional. (This is equivalent to saying that λ is an isolated point of $\sigma(T)\backslash\sigma_e(T)$.) Let $\sigma_0(T)$ denote the set of all normal eigenvalues of the operator T.

Since (BD) is first category in (QD), the following is, perhaps, the best possible result that we can expect, along the lines of [**1**, **7**].

THEOREM. *The subset of all those T in (QD) such that*

(i) $\operatorname{interior}\sigma(T) = \phi$;

(ii) $\dim\mathcal{H}(T;\lambda) = 1$ *for all* $\lambda \in \sigma_0(T)$, *and* $\mathcal{H} = \bigvee\{\mathcal{H}(T;\lambda): \lambda \in \sigma_0(T)\}$;

(iii) T *is irreducible; and*

(iv) *given* ε $(0 < \varepsilon < 1)$, *there exists* $B \in (BD)$ *and* $K_\varepsilon \in \mathcal{K}(\mathcal{H})$, *with* $\|K_\varepsilon\| < \varepsilon$, *such that*

$$T = (1 + K_\varepsilon)B(1 + K_\varepsilon)^{-1},$$

is a G_δ-dense subset of (QD).

PROOF. *First Step.* Let $\{P_n\}_{n=1}^\infty$ be a fixed increasing sequence of finite rank orthogonal projections, converging strongly to 1. Let $(QD)^n$ be the set of all those T in (QD) satisfying the following conditions

$(\text{i})_n$ $\operatorname{interior}\sigma(T)$ does not include a disk of radius $1/n$, and there exists a finite rank Riesz idempotent E (for T) such that

$(\text{ii})_n$ $\|E - E^*\| < 1/n$,

$(\text{iii})_n$ $\sigma(T \mid \operatorname{ran} E)$ consists of exactly $\operatorname{rank} E$ distinct normal eigenvalues, and

$(\text{iv})_n$ $\|P_nEP_n - P_n\| < 1/n$.

By using the upper semicontinuity of the spectrum, and the continuity properties of the Functional Calculus, it is not difficult to check that $(QD)^n$ is an open subset of (QD) (see, e.g., [**5**, Chapter 1]). Therefore,

$$(QD)^0 = \bigcap_{n=1}^{\infty} (QD)^n$$

is a G_δ subset of (QD). A fortiori, so is

$$(QD)^0_{\text{irr}} := \{T \in (QD)^0 : T \text{ is irreducible}\}.$$

It is completely apparent that every T in $(QD)^0_{\text{irr}}$ satisfies (i) and (iii). Moreover, since T satisfies $(\text{ii})_n$, $(\text{iii})_n$ and $(\text{iv})_n$ for all $n = 1, 2, \ldots$, it is not difficult to infer that T admits a sequence $\{E_k\}_{k=1}^\infty$ of finite rank Riesz idempotents such that $E_kE_h = E_hE_k = E_k$ for $1 \le k \le h$, $\sigma(T \mid \operatorname{ran} E_k)$ consists of exactly $\operatorname{rank} E_k$ distinct normal eigenvalues of T, and

$$E_k \to 1 \quad (\textit{strongly}) \quad \text{and} \quad \|E_k - E_k^*\| \to 0 \quad (k \to \infty).$$

It readily follows (as in [**1**, **7**]) that T also satisfies (ii).

If E, F are idempotents such that $EF = FE = E$ (so that $\operatorname{ran} E \subset \operatorname{ran} F$),

$$E = \begin{pmatrix} 1 & E_1 & E_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} \operatorname{ran} E \\ \operatorname{ran} F \ominus \operatorname{ran} E, \\ \mathscr{H} \ominus \operatorname{ran} F \end{array}$$

$$F = \begin{pmatrix} 1 & 0 & F_1 \\ 0 & 1 & F_2 \\ 0 & 0 & 0 \end{pmatrix},$$

and we define

$$V = \begin{pmatrix} 1 & E_1 & E_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then V is invertible,

$$V^{-1} = \begin{pmatrix} 1 & -E_1 & -E_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$VEV^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is the orthogonal projection onto $\operatorname{ran} E$, and

$$VFV^{-1} = \begin{pmatrix} 1 & 0 & F_1 + E_1F_2 - E_2 \\ 0 & 1 & F_2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & F_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

(Indeed, a straightforward computation shows that the condition $EF = FE = E$ is actually equivalent to $E_2 = F_1 + E_1F_2$.)

Thus, if we define

$$W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & F_2 \\ 0 & 0 & 1 \end{pmatrix},$$

then W is invertible,

$$W^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -F_2 \\ 0 & 0 & 1 \end{pmatrix},$$

W and W^{-1} commute with VEV^{-1} (so that

$$(WV)E(WV)^{-1} = W(VEV^{-1})W^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$(VW)F(VW)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is the orthogonal projection onto $\operatorname{ran} F$); moreover,

$$||V-1|| = ||(E_1 E_2)|| = \left\| \begin{pmatrix} 0 & E_1 & E_2 \\ -E_1^* & 0 & 0 \\ -E_2^* & 0 & 0 \end{pmatrix} \right\| = ||E - E^*||$$

and

$$||W-1|| = ||F_2|| = \left\| \begin{pmatrix} 0 & F_2 \\ -F_2^* & 0 \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} 0 & 0 & F_1 \\ 0 & 0 & F_2 \\ -F_1^* & -F_2^* & 0 \end{pmatrix} \right\| = ||F - F^*||.$$

If, in addition, E and F have finite rank, then $V-1$ and $W-1$ are also finite rank operators.

By passing, if necessary, to a subsequence, we can directly assume that $||E_k - E_k^*|| < \varepsilon/2^{k+2}$. By an obvious inductive argument, we can now construct a sequence $\{W_k = V_k V_{k-1} \cdots V_2 V_1\}_{k=1}^\infty$ of invertible operators of the form "1+ Finite rank", such that

$$W_k E_j W_k^{-1} \text{ is the orthogonal projection onto } \operatorname{ran} E_j$$

for $j = 1, 2, \ldots, k$, $||V_k - 1|| \leq ||E_k - E_k^*|| < \varepsilon/2^{k+2}$,

$$\begin{aligned} ||W_k - 1|| &= ||V_k V_{k-1} \cdots V_2 V_1 - 1|| \\ &\leq \sum_{j=1}^{k-1} ||V_k V_{k-1} \cdots V_j - V_k V_{k-1} \cdots V_{j+1}|| \\ &\leq \sum_{j=1}^{k-1} ||V_k|| \cdot ||V_{k-1}|| \cdots ||V_{j+1}|| \cdot ||V_j - 1|| \\ &\leq \sum_{j=1}^{k-1} \left\{ \prod_{r=j+1}^{k} (1 + \varepsilon/2^{r+2}) \right\} \varepsilon/2^{j+2} \\ &\leq \sum_{j=1}^{k-1} \exp \left\{ \sum_{r=j+1}^{k} \varepsilon/2^{r+2} \right\} \varepsilon/2^{j+2} \\ &< \left(\exp \left\{ \sum_{r=2}^{\infty} \varepsilon/2^{r+2} \right\} \right) \sum_{j=1}^{\infty} \varepsilon/2^{j+2} \\ &= e^{\varepsilon/8} \varepsilon/4 < \varepsilon/2 \end{aligned}$$

for all $k = 1, 2, \ldots$, and similarly (for $1 \leq k < h < \infty$),

$$||W_h - W_k|| < e^{\varepsilon/8} \sum_{j=k}^{h-1} \varepsilon/2^{j+2} < e^{\varepsilon/8} \varepsilon/2^{k+1} \to 0 \quad (k, h \to \infty).$$

We conclude that $\{W_k\}_{k=1}^\infty$ is a Cauchy sequence converging in the norm to an invertible operator W of the form

$$W = (1 + K_\varepsilon)^{-1}$$

(where $K_\varepsilon \in \mathscr{K}(\mathscr{H})$ and $||K_\varepsilon|| < \varepsilon$) such that

$$\{F_k = W E_k W^{-1}\}_{k=1}^\infty$$

is an increasing sequence of finite rank orthogonal projections coverging strongly to the identity. (The above construction was loosely based on [**4** and **9**].)

Clearly,

$$B = WTW^{-1}$$

commutes with F_k for all $k = 1, 2, \dots$, whence we infer that $B \in (BD)$ and

$$T = W^{-1}BW = (1 + K_\varepsilon)B(1 + K_\varepsilon)^{-1}$$

has the desired form, that is, T also satisfies (iv).

Second Step. It only remains to show that $(QD)^0_{\mathrm{irr}}$ is dense in (QD).

Claim. $(QD)^0 \cap (BD)$ is dense in (QD).

Let $A \in (QD)$ and let $\eta > 0$ be given. G. R. Luecke proved in [**8**] that invertible quasidiagonal operators are dense in (QD). Let $\{\lambda_k\}_{k=0}^\infty$ be an enumeration of all those complex numbers whose real and imaginary components are rational numbers ($\lambda_0 = 0$). By Luecke's result, there exists $A_0 \in (QD)$ invertible, such that $||A - A_0|| < \eta/2$.

Assume we have already constructed $A_0, A_1, \dots, A_n$ so that $A_k - \lambda_j$ is invertible for $0 \le j \le k \le n$, and

$$||(A_k - \lambda_j)^{-1}||^{-1} > \delta_j > 0$$

$(k = j, j+1, \dots, n; j = 0, 1, 2, \dots, n)$.

By a formal repetition of Luecke's argument, now we can find $A_{n+1} \in (QD)$ such that $A_{n+1} - \lambda_{n+1}$ is invertible, $||(A_{n+1} - \lambda_{n+1})^{-1}||^{-1} = 2\delta_{n+1} > 0$, $||A_{n+1} - A_n|| < \eta/2^{n+1}$, and $A_{n+1} - \lambda_j$ is invertible and satisfies

$$||(A_{n+1} - \lambda_j)^{-1}||^{-1} > \delta_j$$

for $j = 0, 1, 2, \dots, n$.

Clearly, $\{A_n\}_{n=0}^\infty$ is a Cauchy sequence in (QD) converging to a quasidiagonal operator C, which satisfies

$$||A - C|| < \eta \quad \text{and} \quad ||(C - \lambda_k)^{-1}||^{-1} \ge \delta_k > 0$$

for all $k = 0, 1, 2, \dots$. Hence, $\sigma(C) \cap \{\lambda_k\}_{k=0}^\infty = \emptyset$, and therefore

$$\operatorname{interior} \sigma(C) = \phi.$$

According to [**3**], we can find

$$D = \bigoplus_{n=1}^\infty D_n \in (BD)$$

such that $C - D$ is compact and $||C - D|| < \eta$. Here

$$D_n = \begin{pmatrix} d_{11}(n) & & & * \\ & d_{22}(n) & & \\ & & \ddots & \\ 0 & & & d_{r_n r_n}(n) \end{pmatrix} \begin{array}{l} e_1(n) \\ e_2(n) \\ \vdots \\ e_{r_n}(n) \end{array}$$

with respect to some orthonormal basis $\{e_j(n)\}_{j=1}^{r_n}$ of the (finite dimensional) subspace of D_n $(n = 1, 2, \dots)$.

Since $\sigma_e(D) = \sigma_e(C)(\subset \sigma(C))$ has empty interior and D is block-diagonal, we can easily check that

$$\sigma(D) = \sigma_e(D) \cup \Big(\bigcup_{n=1}^{\infty} \{d_{jj}(n): j = 1, 2, \ldots, r_n\}\Big)$$

has empty interior. It is obvious that we can find

$$E = \bigoplus_{n=1}^{\infty} \begin{array}{cc} \begin{pmatrix} d'_{11}(n) & & & * \\ & d'_{22}(n) & & \\ & & \ddots & \\ 0 & & & d'_{r_n r_n}(n) \end{pmatrix} & \begin{array}{c} e_1(n) \\ e_2(n) \\ \vdots \\ e_{r_n}(n) \end{array} \end{array}$$

(The *-entries are exactly the same as for D_n, $n = 1, 2, \ldots$) such that

$$D - E = \bigoplus_{n=1}^{\infty} \begin{array}{cc} \begin{pmatrix} d_{11}(n) - d'_{11}(n) & & & 0 \\ & d_{12}(n) - d'_{22}(n) & & \\ & & \ddots & \\ 0 & & & d_{r_n r_n}(n) - d'_{r_n r_n}(n) \end{pmatrix} & \begin{array}{c} e_1(n) \\ e_2(n) \\ \vdots \\ e_{r_n}(n) \end{array} \end{array}$$

is a trace class diagonal operator such that

$$\begin{aligned} \|D - E\| &= \max\{|d_{jj}(n) - d'_{jj}(n)|: 1 \le j \le r_n,\ n = 1, 2, \ldots\} \\ &\le \sum_{n=1}^{\infty} \sum_{j=1}^{r_n} |d_{jj}(n) - d'_{jj}(n)| = |D - E|_1 < \eta \end{aligned}$$

($|\cdot|_1$ denotes the trace norm) and $\sigma(E)$ is the disjoint union of $\sigma_e(E) = \sigma_e(C)$ and

$$\sigma_0(E) = \bigcup_{n=1}^{\infty} \{d'_{jj}(n): j = 1, 2, \ldots, r_n\};$$

moreover, E can be chosen so that $\mathscr{H}(E;\lambda)$ has dimension one for each $\lambda \in \sigma_0(E)$.

It follows from our construction that

$$E \in (QD)^0 \cap (BD) \quad \text{and} \quad \|A - E\| < 3\eta.$$

Since η can be chosen arbitrarily small, we deduce that $(QD)^0 \cap (BD)$ is dense in (QD).

Therefore, $(QD)^0$ is a G_δ-dense subset of (QD). Since $(QD)_{\text{irr}} = (QD) \cap \{T \in \mathscr{L}(\mathscr{H}): T \text{ is irreducible}\}$ is also a G_δ-dense subset, we conclude that

$$(QD)^0_{\text{irr}} = (QD)^0 \cap (QD)_{\text{irr}}$$

is a G_δ-dense subset of (QD).

The proof of the theorem is now complete. □

REMARK. Since biquasitriangular operators are norm-limits of algebraic operators [**5**, Chapter 6, **11**], and the spectrum of an algebraic operator is finite (and therefore totally disconnected and with empty interior), the same kinds of arguments show that the subsets

$$\{T \in \mathscr{L}(\mathscr{H}): \text{(i) } T \text{ is bitriangular, (ii) } \sigma(T) \text{ is totally disconnected;}$$
$$\text{(iii) } \dim \mathscr{H}(T;\lambda) = 1 \text{ for all } \lambda \in \sigma_0(T) \text{ and } \mathscr{H} = \bigvee\{\mathscr{H}(T;\lambda)\lambda \in \sigma_0(T)\}\}.$$

and

$$\{T \in \mathscr{L}(\mathscr{H})\colon \text{(i) } T \text{ is bitriangular; (ii) interior } \sigma(T) = \phi;$$
$$\text{(iii) } \dim \mathscr{H}(T;\lambda) = 1 \text{ for all } \lambda \in \sigma_0(T) \text{ and } \mathscr{H} = \bigvee\{\mathscr{H}(T;\lambda)\colon \lambda \in \sigma_0(T)\}\}$$

are G_δ-dense subsets of the class of all biquasitriangular operators. (Compare with Theorem 2 and Corollary 9 of [**7**]!)

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