ISOTHERMIC SURFACES AND THE GAUSS MAP

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ABSTRACT. We give a necessary and sufficient condition for the Gauss map of an immersed surface $M$ in $n$-space to arise simultaneously as the Gauss map of an anti-conformal immersion of $M$ into the same space. The condition requires that the lines of curvature of each normal section lie on the zero set of a harmonic function. The result is applied to a class of surfaces studied by S. S. Chern which admit an isometric deformation preserving the principal curvatures.

1. Introduction. The classical Gauss map of a surface in $\mathbb{R}^3$ assigns to a point the unit normal vector to the surface. For a surface in $\mathbb{R}^N$ the Gauss map assigns to a point the tangent plane which may be identified with a point in a quadric $Q^{N-2} \subset CP^{N-1}$.

In recent years several papers have discussed the determination of a surface by its Gauss map. The results of K. Kenmotsu [6] show that a smooth map from a Riemann surface $\mathcal{R}$ to the 2-sphere, satisfying an integrability condition, factors through a conformal immersion

\[(1.1) \quad X: \mathcal{R} \rightarrow M^2 \subset \mathbb{R}^3\]

as the Gauss map. Kenmotsu's integrability condition explicitly involves the conformal structure of $\mathcal{R}$ and a real valued function $h$ which is to be the mean curvature of $M$.

In [3] Hoffman and Osserman give conditions on a map

\[(1.2) \quad g: \mathcal{R} \rightarrow Q^{N-2}\]

involving only the complex structure of $\mathcal{R}$, which are necessary and sufficient for the map to arise as the Gauss map of a conformal immersion

\[(1.3) \quad X: \mathcal{R} \rightarrow \mathbb{R}^N.\]

Their results essentially imply that a conformal immersion of a Riemann surface $\mathcal{R}$ into $\mathbb{R}^N$ is determined by its Gauss map, provided the mean curvature is not identically zero at some point.

Since a single map as in (1.2) determining multiple conformal immersions is in general excluded, a natural question to consider is when such a map arises simultaneously as the Gauss map of both a conformal and anti-conformal immersion. This question turns out to have a simple geometric answer which requires the following definition:

DEFINITION. A surface $M \subset \mathbb{R}^N$ is isothermic if, locally, there exist a pair of harmonic functions $u_1, u_2$ such that the lines of curvature of each smooth section of...
the normal bundle are contained in a level set $u_j = \text{const}$. This is a generalization of a classical definition which can be found in [2].

**THEOREM I.** Let

$$g: \mathbb{R} \to Q^{N-2}$$

be the Gauss map of a conformal immersion of an orientable surface $M$ :

$$X: \mathbb{R} \to M \subset \mathbb{R}^N.$$  

Then there exists an anti-conformal immersion

$$\tilde{X}: \mathbb{R} \to \tilde{M} \subset \mathbb{R}^N$$

such that the following diagram commutes

$$\begin{array}{ccc}
\mathbb{R} & \xrightarrow{X} & Q^{N-2} \\
\downarrow & \searrow & \downarrow \text{Gauss Map} \\
\tilde{X}\downarrow & \tilde{M} & \\
\end{array}$$

if and only if $M$ is isothermic. The surface $\tilde{M}$ is unique (modulo similarity transformations of $\mathbb{R}^N$) provided $M$ is not totally umbilic.

There are abundant examples of isothermic surfaces. We list a few.

1. Constant mean curvature surfaces in $\mathbb{R}^3$.
2. Surfaces of revolution in $\mathbb{R}^3$.
3. Constant mean curvature surfaces $M^2 \subset S^3 \subset \mathbb{R}^4$.

In addition, we note the following properties of isothermic surfaces which when combined with the above furnish more examples:

1. $f(M)$ is isothermic if $M$ is isothermic and $f: \mathbb{R}^N \to \mathbb{R}^N$ is conformal.
2. If $X: \mathbb{R} \to M \subset \mathbb{R}^N$ is isothermic then

$$\underbrace{X \oplus X \oplus \cdots \oplus X}_{k}: M \to \mathbb{R}^{Nk}$$

is isothermic.

In part 4 we apply the main result to a class of surfaces studied by S. S. Chern in [1]. These are surfaces of nonzero mean curvature which admit a nontrivial isometric deformation preserving the principal curvatures. We show first that these surfaces are isothermic and second that the surface $\tilde{M}$ described above has the property that is mean curvature is the reciprocal of a harmonic function.

**2. Preliminaries.** Let $\mathbb{R}$ be a simply connected Riemann surface and

$$X: \mathbb{R} \to M \subset \mathbb{R}^3$$

a smooth, conformal immersion. We assume $M$ is orientable. After choosing a complex coordinate $z = u_1 + iu_2$ on $\mathbb{R}$, the metric induced by (2.1) has an expression

$$ds_M^2 = e^\rho |dz|^2$$

for a smooth function $\rho = \rho(z)$ on $\mathbb{R}$. Locally on $M$ we may choose an orthonormal frame $\{\xi^r\}_{r=3,\ldots,N}$ for the normal bundle $N(M)$. Differentiating $\xi^r$ defines

$$d\xi^r = -A_r(\cdot) + \nabla^\perp(\cdot)\xi^r.$$
The terms on the right-hand side are respectively the tangential and normal components of $d\xi^r$. At each point of $M$, $-A_r$ is a selfadjoint endomorphism of the tangent plane. Its eigenvalues $\beta_j^r; j = 1, 2$ are the principal curvatures of $\xi^r$. The corresponding eigenvectors are the principal directions and their integral curves are the lines of curvature. These curves exist away from the $\xi^r$ umbilics which are those points of $M$ where $\beta_1^r = \beta_2^r$.

The second fundamental forms are defined by

\[(2.4) \quad \Pi^r(X, Y) = ds^2_M(X, A_r(Y)), \quad X, Y \in T_pM.\]

The trace of $\Pi^r$ is twice the $r$th mean curvature

\[(2.5) \quad h^r = \frac{1}{2}(\beta_1^r + \beta_2^r).\]

On $R$, $\Pi^r$ has an expression

\[(2.6) \quad \Pi^r = 2 \text{Re} \left( \frac{\phi^r}{2} dz \otimes dz + \frac{h^r}{2} d\bar{z} \otimes d\bar{z} \right).\]

The quantities

\[(2.7) \quad q^r \equiv \phi^r dz \otimes dz\]
define invariant quadratic differentials on $M$. Under change of complex coordinate

\[(2.9) \quad z_1 = z_1(z), \quad \frac{dz_1}{d\bar{z}} = 0,\]

$q^r$ transforms according to

\[(2.10) \quad q^r = \phi^r dz \otimes dz = \phi^r \left( \frac{dz}{dz_1} \right)^2 dz_1 \otimes dz_1;\]

that is

\[(2.11) \quad \phi^r_1 = \phi^r \left( \frac{dz}{dz_1} \right)^2.\]

The zeros of $q^r$ are precisely the $\xi^r$ umbilics. See [5] for details.

The quantities defined above appear as coefficients of the structural equations for the immersion

\[\begin{cases}
X_{zz} = \rho_z X_z + \frac{1}{2} \sum_r \phi^r \xi^r, \\
X_{z\bar{z}} = \frac{e^\rho}{2} \sum_r h^r \xi^r, \\
\xi_z = -h^r X_z - \phi^r e^{-\rho} X_{\bar{z}} + \sum_t S_t^r \xi^t,
\end{cases}\]

(and their conjugates) where $S_t^r$ are defined by

\[(2.13) \quad \nabla_{\bar{z}}^1 \xi^r = \sum_{t=3}^N S_t^r \xi^t.\]

The well-known integrability conditions of (2.12) are the Gauss equation

\[(2.14) \quad \rho_{z\bar{z}} = \frac{1}{2} \sum_r (\phi^r \bar{\phi}^r e^{-\rho} - (h^r)^2 e^\rho),\]
the Codazzi equations
\[(2.16) \quad \phi^r_x + \sum_i \phi^i \phi_i^r = e^\rho (h^r)_x + \sum_i e^\rho h^i S_i^r \]
and the Ricci equations
\[(2.17) \quad \text{Im} \left\{ (S_i^r)_x - \frac{e^{-\rho}}{2} \phi^r \phi^i + \sum_{l=3}^N S_i^r S_l^i \right\} = 0, \quad 3 \leq r, t \leq N. \]

We define the Gauss map of
\[(2.18) \quad X: \mathbb{R}^2 \to M \subset \mathbb{R}^N \]
following [2]. Consider the quadratic
\[Q^{N-2} = \{ [\zeta] \in \mathbb{C}P^{N-1} | \zeta \cdot \zeta = 0 \}. \]
Here \([ \ ]\) denotes equivalence class. Since \(X\) is conformal, one has
\[(2.19) \quad X_x \cdot X_x = 0 \]
and we define the Gauss map
\[(2.20) \quad g: M \to Q^{N-2}, \quad p \mapsto [X_z], \]
where \(p\) is a point on \(M\) with coordinate \(z\). We will also use \(g\) to denote this map pulled back to \(R\) via \(X\). Many details and interesting properties of \(g\) may be found in [3 and 4].

3. Main result. The proposition below gives simple coordinate-dependent criteria for a surface to be isothermic.

PROPOSITION 3.1. \(M\) is isothermic if and only if locally there exists an isothermal parameter \(z = u + iu_2\) with
\[(3.1) \quad \text{Im} \phi^r (z) = 0, \quad r = 3, \ldots, N. \]

PROOF. Assume (3.1) holds and write
\[(3.2) \quad \Pi^r = \sum_{i,j=1,2} L_{i,j} \, du_i \otimes du_j. \]
An easy computation shows
\[(3.3) \quad \phi^r = \frac{L_{11} - L_{22}}{2} - iL_{12} \]
so that (3.2) implies
\[(3.4) \quad \Pi^r = L_{11}^r \, du_1 \otimes du_1 + L_{22}^r \, du_2 \otimes du_2 \]
and the lines of curvature are the coordinate curves.

On the other hand suppose \(u_1, u_2\) are harmonic functions such that the lines of curvature are contained in a level set \(\{ u_j = \text{const.}\} \). Since the lines of curvature intersect orthogonally, \(u_1\) and \(u_2\) are harmonic conjugates. Define
\[(3.5) \quad z_1 = u_1 \pm iu_2, \]
the sign chosen so that $z_1 = z_1(z)$ is holomorphic. The lines of curvature are the solutions of

\[(3.6) \quad \text{Im} \phi^* dz^2 = 0\]

(see [5] for details). Since $\nabla u_1$ (resp. $\nabla u_2$) is tangent to the level curves $u_2 = \text{const.}$ (resp. $u_1 = \text{const.}$), (3.6) implies

\[(3.7) \quad \text{Im} \phi^* dz \otimes dz(\nabla u_j, \nabla u_j) = 0.\]

Using

\[(3.8) \quad \nabla u_j = 2e^{-\rho}(u_j \bar{z} X_z + u_j z X_{\bar{z}}), \quad j = 1, 2,\]

this gives

\[(3.9) \quad \text{Im} \phi(u_j \bar{z})^2 = 0, \quad j = 1, 2.\]

The Cauchy-Riemann equations applied to $u_1, u_2$ give

\[(3.10) \quad u_{1\bar{z}} = -iu_{2\bar{z}}\]

and we find

\[
0 = \text{Im} \phi^* \cdot \left( u_{1\bar{z}}^2 - u_{2\bar{z}}^2 \pm 2u_{1\bar{z}} \right) \\
= \text{Im} \phi^* \cdot \left( u_{1\bar{z}}^2 - u_{2\bar{z}}^2 \mp 2iu_{1\bar{z}}u_{2\bar{z}} \right) \\
= \text{Im} \phi^* \cdot (u_{1\bar{z}} \mp iu_{2\bar{z}})^2 \\
= \text{Im} \phi^* \left( \frac{dz}{dz_1} \right)^2 \left| \frac{dz_1}{dz} \right|^2.
\]

Therefore

\[(3.11) \quad \text{Im} \phi^* \left( \frac{dz}{dz_1} \right)^2 = 0\]

and by Proposition (3.1) the coordinate $u_1 \pm iu_2$ is as required.

**Proof of Theorem I.** Let

\[(3.12) \quad g: R \rightarrow Q^{N-2} \subset CP^{N-1}\]

represent the Gauss map of

\[(3.13) \quad X: R \rightarrow M \subset R^N.\]

The existence of $\tilde{X}$ is equivalent to the existence of a smooth $\mathbb{C}$-valued function $f$ on $R$ such that

\[(3.14) \quad e^f X_z = \tilde{X}_{\bar{z}}\]

i.e.,

\[(3.15) \quad [\tilde{X}_{\bar{z}}] = [X_z] \in Q^{N-2}.\]

The differential equation (3.14) for $\tilde{X}$ will be integrable provided

\[(3.16) \quad \text{Im}(\partial_z e^f X_z) = 0.\]
Using (2.12) this becomes

\[ 0 = \text{Im} \left\{ e^f (f_z + \rho_z) X_z + e^f \sum_r \frac{1}{2} \phi^r \xi^r \right\} \]

\[ = \text{Im} \left\{ (f_z + \rho_z) \tilde{X}_z + e^f \sum_r \frac{1}{2} \phi^r \xi^r \right\}. \]

Note that the structural equations (2.12) applied to the immersion \( \tilde{X} \) imply

(3.17) \[ f_z + \rho_z = 0 \]

so we can write

(3.18) \[ f = \bar{g} - \rho \]

with \( g_\xi = 0 \). (\( g \) is holomorphic.) This gives for all \( r \),

(3.19) \[ \text{Im} e^{\bar{g}} \phi^r = 0. \]

Define a new isothermal coordinate

(3.20) \[ \tilde{z}_1 = \int^z e^{g/2} d\xi. \]

Then the transformation rule (2.11) for \( \phi^r \) gives

(3.21) \[ \phi^r_1 = e^{-g} \phi^r \]

and so (3.19) implies

(3.22) \[ \text{Im} \phi^r_1 = 0 \]

and \( M \) is isothermic.

Conversely if \( \text{Im} \phi^r = 0 \) for all \( r \) then define

(3.23) \[ \tilde{X}_\xi = e^{-\rho} X_z \]

and one easily checks using (2.12) that

(3.24) \[ \text{Im} \partial_z (e^{-\rho} X_z) = \text{Im} \left(e^{-\rho(1/2)} \sum_r \phi^r \xi^r \right) = 0. \]

For the uniqueness of \( \tilde{M} \), note that by [4] Theorem 2.3 the Gauss map of \( \tilde{M} \),

(3.25) \[ \tilde{g} = [\tilde{X}_\xi] = [e^f X_z] \]

determines the immersion \( \tilde{X} \) (mod similarities) provided \( \tilde{X}_{\xi z} \neq 0 \). However,

\[ \tilde{X}_{\xi z} = e^f \left((f_z + \phi_z) X_z + \sum_r \frac{\phi^r}{2} \xi^r \right). \]

The right-hand side cannot vanish identically unless \( \phi^r \equiv 0, r = 3, \ldots, N \), which is the totally umbilic case.
4. Application to a class of surfaces. In [1] S. S. Chern classified the umbilic free surfaces $M \subset \mathbb{R}^3$ which admit a nontrivial isometric deformation preserving the principal curvatures. Besides the classical examples of constant mean curvature surfaces, Chern found a second class of Weingarten surfaces with the property that the conformal metric

$$ds^2 = \|\nabla h\|^2(h^2 - K)^{-1}ds_M^2$$

has constant Gaussian curvature $\hat{K} = -1$. We will show these surfaces are isothermic and show the surfaces $\tilde{M}$ obtained by Theorem I have an interesting property.

**THEOREM II.** Let $M$ be an umbilic free, nonminimal, surface in $\mathbb{R}^3$ admitting a nontrivial isometric deformation preserving the principal curvatures. Then

(i) $M$ is isothermic;

(ii) If $\tilde{M}$ is the surface obtained via Theorem I, the mean curvature $\tilde{h}$ of $\tilde{M}$ satisfies

$$\tilde{\Delta}\left(\frac{1}{\tilde{h}}\right) = 0 \quad (\tilde{\Delta} \text{ is the Laplace-Beltrami operator on } \tilde{M}).$$

**REMARK.** If $h \equiv \text{const.}$ on $M$ then the above result is well known. The surface $\tilde{M}$ also has constant mean curvature so (4.1) holds trivially.

Following [1], introduce an orthonormal moving frame $\{e_1, e_2, e_3\}$ along $M$ with $e_1, e_2$ the principal directions and $e_3$ the unit normal. Let $\omega_j$ be the dual one forms and as usual define $\omega_{ij}$ by

$$d\omega_i = \omega_{ij} \wedge \omega_j.$$

By choice of frame we have:

$$\omega_{13} = a\omega_1, \quad \omega_{23} = c\omega_2$$

where $a > c$ are the principal curvatures. Next introduce the one form

$$\theta_1 = \frac{2dh}{a - c}$$

Let $\alpha_1$ be the symmetry of $\theta_1$ with respect to the principal directions:

$$\alpha_1 = \theta_1 - 2\left(\frac{2}{a - c}e_2(h)\right)\omega_2.$$ 

Let $*$ denote the Hodge star operator

$$*\omega_1 = \omega_2, \quad *\omega_2 = -\omega_1$$

and define

$$\theta_2 = *\theta_1, \quad \alpha_2 = *\alpha_1.$$

The Codazzi equations on $M$ can be used to show (see [1])

$$d\log(a - c) = \alpha_1 + 2*\omega_{12}.$$ 

In addition we note the important formulas

$$\begin{align*}
\text{(a)} & \quad d\alpha_1 = 0, \\
\text{(b)} & \quad d\alpha_2 \wedge \alpha_2,
\end{align*}$$

which gives $\hat{K} = -1$ in the case $h \not\equiv \text{const.}$.
Proof of Theorem III. We first show $M$ is isothermic. By (4.9) and (4.10a) we have
\[(4.12) \quad d^*\omega_{12} = 0.\]
Locally we can define a function $\rho$ by
\[(4.13) \quad *\omega_{12} = \frac{-d\rho}{2}.\]
Define
\[(4.14) \quad du_j = e^{-\rho/2}u_j, \quad j = 1, 2.\]
Then (4.13) together with the structural equations (4.2) imply $ddu_j = 0$. The coordinates $u_1, u_2$ have the necessary property of Proposition 3.1 and $M$ is isothermic.

We assume by the remark that $h \neq \text{const}$. Using (3.23) we compute fundamental quantities on $\tilde{M}$ denoted with “tilde”,
\[(i) \quad \tilde{\omega}_1 = e^{-\rho}\omega_1, \quad \tilde{\omega}_2 = -e^{-\rho}\omega_2,\]
\[(ii) \quad \tilde{a} = e^\rho a, \quad \tilde{c} = -e^\rho c,\]
\[(iii) \quad \tilde{h} = e^\rho \left(\frac{a - c}{2}\right).\]

The Hodge star operator of $\tilde{M}$ gives
\[(4.16) \quad \tilde{\omega}_1 = \omega_2, \quad \tilde{\omega}_2 = -\omega_1\]
which gives using (4.15(i))
\[(4.17) \quad \tilde{\omega}_1 = -\omega_2, \quad \tilde{\omega}_2 = \omega_1.\]
By (4.15(iii)), equation (4.1) is equivalent to
\[0 = d^*d(e^{-\rho}(a - c)^{-1}).\]
Compute
\[d^*d(e^{-\rho}(a - c)^{-1}) = -d^* e^{-\rho}[(a - c)^{-1}d\rho - (a - c)^{-1}d\log(a - c)]\]
\[= -d^* e^{-\rho}[(a - c)^{-1}d\rho - (a - c)^{-1}(\alpha_1 - d\rho)] \quad \text{(by (4.9), (4.13))}\]
\[= d^* e^{-\rho}(a - c)^{-1}\alpha_1\]
\[= -d e^{-\rho}(a - c)^{-1}\alpha_2 \quad \text{(by (4.5), (4.18))}\]
\[= e^{-\rho}[(a - c)^{-1}d\rho \wedge \alpha_2 + (a - c)^{-1}d\log(a - c) \wedge \alpha_2 + (a - c)^{-1}d\alpha_2] \quad \text{(by (4.9), (4.10b))}\]
\[= e^{-\rho}[(a - c)^{-1}d\rho \wedge \alpha_2 + (a - c)^{-1}d \cdot (\alpha_1 - d\rho) \wedge \alpha_2 - (a - c)^{-1}\alpha_1 \wedge \alpha_2]\]
\[= 0.\]
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