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AN ISOPERIMETRIC THEOREM ON THE CUBE AND THE KINTCHINE-KAHANE INEQUALITIES

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ABSTRACT. For vectors $x_1, \ldots, x_n$ in a Banach space, we bound the deviation of $\|\sum_{i\leq n} \varepsilon_i x_i\|$ from its median.

1. Results. Consider the set $\{-1,1\}^n$, provided with its canonical probability measure P_n that gives mass 2^{-n} to each point. We will consider $\{-1,1\}^n$ as a subset of the n-dimensional Hilbert space $\mathbf{R}^n$. For a nonempty subset of A of $\{-1,1\}^n$, we set $\phi_A(x) = \inf\{\|x-y\|_2; y \in \operatorname{conv} A\}$.

THEOREM 1. $E\exp(\phi_A^2/8) \leq 1/P_n(A)$.

From Chebyshev's inequality we get

COROLLARY 2. *For all* $t \geq 0$, *we have* $P_n(\{\phi_A \geq t\}) \leq (1/P_n(A))e^{-t^2/8}$.

To understand this result, it might be helpful to compare it with classical results concerning the Hamming distance. The Hamming distance $d(s,t)$ between $s,t \in \{0,1\}^n$ is the number of coordinates where s and t differ. For a subset A of $\{-1,1\}^n$, we set $d_A(x) = \inf\{d(x,y); y \in A\}$. The largest possible value of $P_n(\{d_A \geq t\})$ when $P_n(A)$ is given is known. The sets for which $P_n(\{d_A \geq t\})$ is maximum are identified in [**3**]. In particular, when $P_n(A) = 1/2$, it is known (see [**1**]) that

$$P_n(\{d_A \geq t\sqrt{n}\}) \leq \tfrac{1}{2}e^{-2t^2}. \tag{1-1}$$

To make the connection with Corollary 2, we observe the following.

Fact. $2d_A \leq \sqrt{n}\phi_A$.

PROOF. Fix x, and consider the linear functional θ on $\mathbf{R}^n$ given by

$$\theta(z) = \sum_{i\leq n} n^{-1/2} x_i z_i.$$

It is of norm 1. For any $y \in A$, we have

$$\theta(x-y) = \sum_{i\leq n} \frac{1}{\sqrt{n}} x_i(x_i - y_i) = \frac{2}{\sqrt{n}} d(x,y) \geq \frac{2}{\sqrt{n}} d_A(x).$$

It follows that for each $y \in \operatorname{conv} A$, we have $\theta(x-y) \geq (2/\sqrt{n})d_A(x)$ so that $\|x-y\|_2 \geq (2/\sqrt{n})d_A(x)$, which proves the fact.

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The fact shows that

$$\{d_A \geq (t/2)\sqrt{n}\} \subset \{\phi_A \geq t\}. \tag{1-2}$$

While (1-1) provides the estimate

$$P_n(\{d_A \geq (t/2)\sqrt{n}\}) \leq \tfrac{1}{2}e^{-t^2/2},$$

Corollary 2 provides a larger bound for a larger set:

$$P_n(\{\phi_A \geq t\}) \leq 2e^{-t^2/8}. \tag{1-3}$$

Our method of proof cannot give in Theorem 1 a coefficient better than 1/8. It is to be noted however that (1-3) is better than the estimates for $P_n(\{d_A \geq t\sqrt{n}/2\})$ that one obtains by martingale methods (see [**4**, p. 36]). We do not know the answer to the following.

Problem. What is the value of

$$\max\{P_n(\{\phi_A(x) \geq t\}); P_n(A) = a\}?$$

This, and a related problem, is discussed in §3.

THEOREM 3. *Consider a* convex *Lipschitz function f on $(\mathbf{R}^n, \|\cdot\|_2)$, of Lipschitz constant σ. Denote by M a median of f (for P_n). Then we have for all $t \geq 0$,*

$$P_n(\{|f - M| > t\}) \leq 4e^{-t^2/8\sigma^2}.$$

An interesting feature of this result is that it is really specific to convex functions. To see this, let $A = \{x \in \{-1,1\}^n : \sum_{i\leq n} x_i \leq 0\}$ and define $f(x) = \inf\{\|x-y\|_2 : y \in A\}$, so that f has a Lipschitz constant 1, and $M = 0$. It is easy to see that for $y \in \{-1,1\}^n$, we have $f(y)/\sqrt{2} = d(y,A)^{1/2} = ((\sum_{i\leq n} y_i)^+)^{1/2}$. It follows from the central limit theorem that $P_n(\{f > cn^{1/4}\}) \geq 1/4$ for some constant c independent of n.

COROLLARY 4. *Consider a Banach space E, and $(x_i)_{i\leq n}$ in E. Let*

$$\sigma^2 = \sup\left\{\sum_{i\leq n} x^*(x_i)^2; x^* \in E^*, \|x^*\| \leq 1\right\}. \tag{1-4}$$

Consider an i.i.d. sequence $(\varepsilon_i)_{i\leq n}$ of Bernoulli random variables. Let M be a median of $\|\sum_{i\leq n}\varepsilon_i x_i\|$. Then for all $t \geq 0$,

$$P\left(\left\{\left|\left\|\sum_{i\leq n}\varepsilon_i x_i\right\| - M\right| \geq t\right\}\right) \leq 4e^{-t^2/8\sigma^2}. \tag{1-5}$$

This inequality should be compared with the inequality

$$P\left(\left\{\left|\left\|\sum_{i\leq n} g_i x_i\right\| - M\right| \geq t\right\}\right) \leq \frac{2}{\sigma\sqrt{2\pi}}\int_t^\infty e^{-u^2/2\sigma^2}\,du \leq 2e^{-t^2/2\sigma^2}, \tag{1-6}$$

where the second inequality holds for $t \geq \sigma(2\pi)^{1/2}$, and where $(g_i)_{i\leq n}$ are i.i.d. $N(0,1)$ and M is now the median of $\|\sum_{i\leq n} g_i x_i\|$. (Inequality (1-6) is a well-known consequence of Borell's isoperimetric inequality [**2**].) Inequality (1-5) allows one to simplify (at least conceptually) the theory of probability in a Banach space, as it often allows one to replace Gaussian randomization by Bernoulli randomization.

By a standard computation, we deduce from Corollary 4 the following version of the Kintchine-Kahane inequalities.

COROLLARY 5. *There exists a universal constant K, such that for any elements $x_1, \ldots, x_n$ of a Banach space, we have, for $p \geq 1$*

$$\left\|\sum_{i\leq n} \varepsilon_i x_i\right\|_p \leq \left\|\sum_{i\leq n} \varepsilon_i x_i\right\|_1 + K\sigma p^{1/2}$$

where σ is given by (1-4).

2. Proofs.

PROOF OF THEOREM 1 We first consider the case where $\operatorname{card} A = 1$. Then

$$E\exp\left(\frac{\phi_A^2}{8}\right) = 2^{-n}\sum_{0\leq i\leq n}\binom{n}{i}e^{i/2} = \left(\frac{1+e^{1/2}}{2}\right)^n \leq 2^n = \frac{1}{P_n(A)}$$

since $e^{1/2} < e < 3$.

Next, we prove Theorem 1 when $n = 1$. It remains only to consider the case where $A = \{-1, 1\}$, so that $\phi_A(x) \equiv 0$, and the result holds.

We now prove Theorem 1 by induction over n. Assuming it holds for n, we prove it for $n+1$. It is enough to consider the case where A has at least 2 points. Identifying $\{-1,1\}^{n+1}$ with $\{-1,1\}^n \times \{-1,1\}$, we can suppose that $A = A_{-1} \times \{-1\} \cup A_1 \times \{1\}$ where $A_{-1}, A_1 \neq \emptyset$, $A_{-1}, A_1 \subset \{-1,1\}^n$. For definiteness we assume that $P_n(A_{-1}) \leq P_n(A_1)$. We observe that for $x \in \{-1,1\}^n$, we have

$$\phi_A((x,1)) \leq \phi_{A_1}(x). \tag{2-1}$$

Fact. For $x \in \{-1,1\}^n$, and $0 \leq \alpha \leq 1$, we have

$$\phi_A^2((x,-1)) \leq 4\alpha^2 + \alpha\phi_{A_1}^2(x) + (1-\alpha)\phi_{A_{-1}}^2(x). \tag{2-2}$$

PROOF OF FACT. For $i = -1, 1$, let $z_i \in \operatorname{conv} A_i$ such that $\|x - z_i\|_2 = \phi_{A_i}(x)$. We observe that $(z_i, i) \in \operatorname{conv} A$, so that $z = (\alpha z_1 + (1-\alpha)z_{-1}, -1+2\alpha) \in \operatorname{conv} A$. Now

$$\begin{aligned}\|(x,-1) - z\|_2^2 &= 4\alpha^2 + \|x - (\alpha z_1 + (1-\alpha)z_{-1})\|_2^2 \\ &= 4\alpha^2 + \|\alpha(x - z_1) + (1-\alpha)(x - z_{-1})\|_2^2 \\ &\leq 4\alpha^2 + \alpha\|x - z_1\|_2^2 + (1-\alpha)\|x - z_{-1}\|_2^2\end{aligned}$$

by the triangle inequality and the convexity of t^2. This proves the fact.

For $i = -1, 1$, we set $u_i = E\exp(\phi_{A_i}/8)$, and $v_i = 1/P(A_i)$, so that $u_i \leq v_i$ by the induction hypothesis. From (2-1), (2-2), we have for all $0 \leq \alpha \leq 1$,

$$\begin{aligned}E\exp(\phi_A^2/8) &\leq \tfrac{1}{2}E\exp(\phi_{A_1}^2/8) \\ &\quad + \tfrac{1}{2}E\exp(\alpha^2/2 + \alpha\phi_{A_1}^2/8 + (1-\alpha)\phi_{A_{-1}}^2/8) \\ &\leq \tfrac{1}{2}u_1 + \tfrac{1}{2}e^{\alpha^2/2}u_1^{\alpha}u_{-1}^{1-\alpha} \\ &\leq \tfrac{1}{2}v_1[1 + e^{\alpha^2/2}(v_1/v_{-1})^{\alpha-1}]\end{aligned}$$

by Hölder's inequality and since $u_i \leq v_i$. The value of α that minimizes the above expression is $\alpha = -\log(v_1/v_{-1})$, but, in order not to have to consider the case where $\alpha \geq 1$, we take $\alpha = 1 - v_1/v_{-1}$, which gives

$$E\exp(\phi_A^2/8) \leq \tfrac{1}{2}v_1[1 + e^{\alpha^2/2}(1-\alpha)^{\alpha-1}].$$

LEMMA. *If* $0 \leq \alpha < 1$, *we have*

$$1 + e^{\alpha^2/2}(1-\alpha)^{\alpha-1} \leq 4/(2-\alpha).$$

Indeed, this is equivalent to saying that

$$e^{\alpha^2/2}(1-\alpha)^{\alpha-1} \leq (2+\alpha)/(2-\alpha)$$

or that

$$\alpha^2/2 - (1-\alpha)\log(1-\alpha) \leq \log(1+\alpha/2) - \log(1-\alpha/2).$$

But this is easily seen by consecutive differentation. □

We now have

$$E\exp(\phi_a^2/8) \leq \frac{1}{2}v_1\left(\frac{4}{2-\alpha}\right) = v_1\left(\frac{2}{1+v_1/v_{-1}}\right) = \frac{2}{1/v_1 + 1/v_{-1}}$$
$$= \frac{2}{P_n(A_1) + P_n(A_{-1})} = \frac{1}{P_{n+1}(A)}.$$

This finishes the proof of Theorem 2.

PROOF OF THEOREM 3. By definition of a median, we have

$$P_n(\{f \geq M\}) \geq 1/2, \qquad P_n(\{f \leq M\}) \geq 1/2.$$

Let $A = \{f \leq M\}$. Since f is convex, we have $f \leq M$ on $\operatorname{conv} A$. Since f has a Lipschitz constant σ, we have $f(x) \geq M + t \Rightarrow \phi_A(x) \geq t/\sigma$. Hence, by Corollary 2,

$$P_n(\{f \geq M + t\}) \leq P_n(\{\phi_A \geq t/\sigma\}) \leq 2e^{-t^2/8\sigma^2}.$$

Now let $t > 0$, and consider the set $B = \{f \leq M - t\}$. Let $u < t$. We see as before that

$$f(x) \geq M - t + u \Rightarrow \phi_B(x) \geq u/\sigma,$$

so, by Corollary 2, we have

$$1/2 \leq P_n(\{f \geq M - t + u\}) \leq (1/P_n(B))e^{-u^2/8\sigma^2}$$

and $P_n(B) \leq 2e^{-u^2/8\sigma^2}$. Letting $u \to t$, we have $P_n(B) \leq 2e^{-t^2/8\sigma^2}$. This proves Theorem 3.

To prove Corollary 4, we apply Theorem 3 to the function $f\colon u \to \|\sum_{i\leq n} u_i x_i\|$ on $\mathbf{R}^n$. This function is obviously convex. Now, for u, $v \in \mathbf{R}^n$, $x^* \in E^*$, $\|x^*\| \leq 1$, we have

$$\left|x^*\left(\sum_{i\leq n} u_i x_i\right) - x^*\left(\sum_{i\leq n} v_i x_i\right)\right| \leq \left|\sum_{i\leq n}(u_i - v_i)x^*(x_i)\right|$$
$$\leq \left(\sum_{i\leq n} x^*(x_i)^2\right)^{1/2} \|u - v\|_2$$

so that $|f(u) - f(v)| \leq \sigma\|u - v\|_2$ and f has a Lipschitz constant σ.

3. Some problems of combinatorics. The first problem is, as mentioned in the introduction, the determination of the function

$$h(a,t) = \max\{P_n(\{\phi_A \geq t\}); P_n(A) = a\}.$$

This problem seems difficult. The reason is that the extremal sets (i.e. the sets for which $P_n(\{\phi_A \geq t\}) = h(a,t)$, $P_n(A) = a$) depend on t. To see it, we note that for $a = 1/2$, the set $A = \{x; x_1 = -1\}$ is extremal for $t < 2$ (since $P_n(\{\phi(x,A) \geq t\}) = 1/2$ is as large as possible) but is not extremal for $t > 2$. Another related problem is as follows. Suppose that we are given a set $A \subset \{-1,1\}^n$, and for each $x \in A$ an element $f_x = (f_{x,i})_{i \leq n}$ of $\mathbf{R}^n$, of norm one and of positive components. We can consider the set

$$B = \left\{ y \in \{-1,1\}^n,\ \forall x \in A,\ \sum_{i \leq n} |y_i - x_i| f_{x,i} > t \right\}.$$

Knowing $P_n(A)$, how large can $P_n(B)$ be? We note that in the case where $f_{x,i} = n^{-1/2}$ for all $x \in A$, all $i \leq n$, then $B = \{y\colon d(x,A) > t\sqrt{n}/2\}$ and the answer is known in that case. In the general case, Corollary 2 implies the bound $P_n(B) \leq (1/P_n(A))e^{-t^2/8}$. Indeed, since $|y_i - x_i| = x_i(x_i - y_i)$, we have $\phi_B(x) \geq t$ on A, so by Corollary 2, we have $P_n(A) \leq (1/P_n(B))e^{-t^2/8}$.

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