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d-FINAL CONTINUA

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ABSTRACT. The Hilbert cube is known to be an irreducible quotient of every perfect metric space. Its irreducible quotients are identified: all nondegenerate Peano continua no open set in which is homeomorphic with R. In compact metric spaces, every irreducible surjection $X \to Y$ embeds a dense G_δ subset of X in Y. The Peano continua I which are strongly initial, every irreducible map from a Peano continuum to I being a homeomorphism, are identified: the dendrites the closure of whose end points is zero-dimensional.

Introduction. Every nondegenerate Peano continuum is a quotient of every other nondegenerate Peano continuum—this easy consequence of the Hahn-Mazurkiewicz theorem serves as a reference point for locating the subject of this paper, which is the quasi-ordering by *irreducible* quotient. The subject was broached by Euler, who determined which finite graphs are irreducible quotients of an arc. The next contributor was Peano, who produced irreducible surjections from an arc to an n-cell for each n. The next would seem to be H. Enos, who showed that the Hilbert cube is an irreducible quotient of everything (more precisely, of any Hausdorff compactification of the rationals [**2**]). Like Euler and Peano, Enos did not mention irreducibility. He referred to continuous maps $X \to Y$ which take some dense subspace of X homeomorphically to a dense subspace of Y. The first result of the present paper is that in compact metrizable spaces, these notions are equivalent; an irreducible surjection restricts to an embedding (necessarily dense) of some dense subspace.

By Lavrentiev's theorem, there is actually a dense G_δ subspace—"almost all of X"—mapped homeomorphically to a dense G_δ subspace of the codomain. In fact the proof of the first result is a pure application of "abstract nonsense": for trivial reasons, an irreducible surjection maps a dense nonspace, a *sublocale* of X, invertibly ["homeomorphically"], and Lavrentiev's theorem is true of locales.

Accordingly irreducible surjections of compact metrizable spaces (and suitable maps of other spaces) are called *d-equivalences* here. All nondegenerate Peano continua are d-equivalent, by [**2**]. But the result is in fact that the Hilbert cube is *d-final*, i.e. it is the codomain of a d-equivalence from any d-equivalent (compact metric) space. The main result of this paper is in effect that Enos' one necessary condition for d-finality of a Peano continuum Y, that the set of local cut points be of the first category, is sufficient. In particular, a 2-cell I^2 is d-final; there exists a continuous map $I^3 \to I^2$ which maps a dense (G_δ) subset of I^3 homeomorphically

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to a dense subset of I^2. (The condition for d-finality is, more simply, that no open subset is homeomorphic with R.)

The main result is proved by studying irreducible maps to and from trees. (The crux is that a compact metric tree, or "dendrite", with dense end points is an irreducible quotient of everything.) Trees T whose set of end points has zero-dimensional closure and which have no cut point of infinite order are what may be called *isolated d-initial*: every irreducible surjection from a Peano continuum to T is invertible. No other Peano continua have that property, but I do not know about mere initiality (minimality in the quasi-ordering).

1. d-equivalences. The question when there is a continuous map $X \to Y$ extending a homeomorphism of some dense subspaces seems to have been first raised by S. Mrowka [**12**], who asked whether there is such a map $f\colon R^2 \to R$. The first result was H. Enos' [**2**], in effect answering "no" to Mrowka's question. Enos considered compact spaces, and omitted the exercise of showing that $f\colon R^2 \to R$ would induce a compact example. (Suppose f maps D, dense in R^2, homeomorphically to E, dense in R. We may assume D and E are countable. Then some circle S in R^2 misses D. D meets the inside and outside of S in open-closed subspaces, and one easily checks that the restriction of f to the disk K bounded by S is a similar map of K to an interval—and does not exist [**2**].)

Enos' result is that these maps, among Peano continua, preserve the property "the set of local cut points is of the first category".

I have no specific information on these maps outside compact metric spaces; but it turns out that (1) in compact metric spaces, they coincide with the familiar class of irreducible continuous surjections; but (2) generalizing in the wake of Gleason, we get a very natural class of maps between arbitrary sober topological spaces. First extend to compact Hausdorff spaces. Irreducible continuous surjections $X \to Y$ no longer need induce a homeomorphism of dense subspaces, but they coincide with maps under a homeomorphism of Gleason (projective) covers. (That is,

$$\begin{array}{ccc} G & \longrightarrow & G' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

commutes, where the verticals are Gleason covers and $G \to G'$ is invertible.) But in locales (a.k.a. 'pointless spaces'), the Gleason cover of a compact Hausdorff space X is just the Stone-Čech compactification of a distinguished *sublocale* $D(X)$: the smallest dense sublocale.

(This fact—in a more general version, farther from topological spaces—is in [**7**, 3.1]. To prove the present assertion, just observe that the Gleason cover is the Stone space of the complete Boolean algebra $B(X)$ of regular open sets, while $D(X)$ is the locale defined by the frame $B(X)$.)

A basic reference for locales is [**6**]. I find the main facts on the dense part $D(X)$ ($X_{\neg\neg}$ in [**6**]) easier to locate in the original paper [**4**].

D is not a functor; a constant map $X \to Y$, for instance, maps $D(X)$ also to one point, which is not in $D(Y)$ unless it is an open point. However, it makes sense to call a map $f\colon X \to Y$ of locales a *d-equivalence* if it induces an isomorphism between $D(X)$ and $D(Y)$. In other words, the image of the sublocale $D(X)$ is contained in $D(Y)$, and the induced map $D(X) \to D(Y)$ is invertible. It is very much like considering homomorphisms of groups which extend isomorphisms of their centers.

The first results are

1.1. *In strongly Hausdorff spaces, a closed d-equivalence is an irreducible surjection.*

1.2. *In sober spaces, a closed irreducible surjection is a d-equivalence.*

The restriction to sober spaces is in effect grammar rather than geometry; only sober spaces can be identified with (their) locales, so that d-equivalences are defined. *Strongly Hausdorff* spaces are those X for which the diagonal in the locale $X \times X$ is closed; they include all regular spaces, but not all Hausdorff spaces. It is in just these spaces that the equalizer of any two continuous maps is closed. In a merely Hausdorff space, the set of points of an equalizer is closed.

PROOF OF 1.1. The image of such a map $f\colon X \to Y$ is closed and contains $D(Y)$, so f is surjective. We have $f|D(X)$ mapping invertibly to $D(Y)$, inverted by $e\colon D(Y) \to D(X)$. Now in $f^{-1}(D(Y))$, $D(X)$ is a dense sublocale. It is also closed, for it is the equalizer of $ef|f^{-1}(D(Y))$ and the identity. So $f^{-1}(D(Y))$ is just $D(X)$. No closed proper subspace K contains dense $D(X)$, so $f(K)$ does not contain $D(Y)$. Thus $f(K) \neq Y$.

PROOF OF 1.2. The inverse image of dense open $U \subset Y$ under such a map $f\colon X \to Y$ is open, and any closed subspace containing it maps onto $U^- = Y$, so $f^{-1}(U)$ is dense. Generally $D(Z)$ is the intersection of all dense open sublocales, and on sublocales, f^{-1} preserves intersections [4], so $D(X)$ maps into $D(Y)$. It maps onto $D(Y)$, i.e. into no proper sublocale, since $Y = f(X) = f(D(X)^-) \subset f(D(X))^-$, and $D(Y)$ is the smallest dense sublocale. Since the frame of $D(X)$ is a Boolean algebra, it remains only to show that disjoint sublocales U_0, V_0 of $D(X)$ have disjoint images. All sublocales of $D(X)$ are open. So in X, U_0^- and V_0 are disjoint, so also U_0^{-0} and V_0, hence U_0^{-0} and V_0^- and finally $U = U_0^{-0}$ and $V = V_0^{-0}$. It is not possible that $f(U) \cap f(V)$ is dense in an open set $W \neq 0$, for then, removing $U \cap f^{-1}(W)$ from X, the remaining closed set would have image (1) containing $Y - W$, obviously, and (2) containing W since it contains $f(V)$ and is closed. Therefore $f(U_0) \cap f(V_0) \subset D(Y)$ is 0.

Thus among closed maps of regular spaces, and in particular

1.3. *In compact Hausdorff spaces, d-equivalences are exactly irreducible continuous surjections.*

Or again, exactly the maps $X \to Y$ whose composite with the projective cover of X gives the projective cover of Y.

(Irreducible surjections which are not closed are nothing in particular; for instance, they include all [nonclosed] continuous bijections. Nonclosed d-equivalences need not be surjective, but include all dense embeddings. In strongly Hausdorff spaces, they do have the property that every closed proper part of X maps into a closed proper part of Y, the proof of 1.1 shows.)

1.4. *Any map of locales which extends an isomorphism of dense sublocales is a d-equivalence.*

PROOF. Obvious.

1.5. *Every d-equivalence of completely metrizable spaces extends a homeomorphism of dense G_δ subspaces.*

PROOF. Actually *Lavrentiev's Theorem holds for completely metrizable locales.* (Completely metrizable locales *are* spaces [**4**]; of course their sublocales need not be.) The proof is essentially the same as for spaces. Given an isomorphism of sublocales $f\colon S \to T$ of complete metric spaces $A \supset S$, $B \supset T$, for each $\varepsilon > 0$, the open parts V of B such that V meets T and both V and $f^{-1}(V \cap T)$ have diameter $< \varepsilon$ cover T; so they cover an open sublocale D_ε containing T. The intersection of all D_ε is a G_δ sublocale D_∞ of B; B is a G_δ sublocale of its Stone-Čech compactification; so [**5**, 4.1] D_∞ is a space. Then one can define $g\colon D_\infty \to S^-$ pointwise in the obvious way: each p in $D_\infty = \bigcap D_\varepsilon$ is, for all $\varepsilon > 0$, in ε-small parts $V = V(p, \varepsilon)$ with $f^{-1}(V \cap T)$ ε-small and nonempty, so the $f^{-1}(V \cap T)$ cluster at a unique point $g(p) \in S^-$. This is continuous because those p have neighborhoods V with $f^{-1}(V \cap T)$ arbitrarily small (diameter $\leq 2\varepsilon$). Now in the space S^- consider the subspace C of all points that have neighborhoods U with $g^{-1}(U)$ arbitrarily small; a G_δ, and routinely, $g^{-1}|C$ is a homeomorphism upon a G_δ subspace $D \subset D_\infty$.

2. Trees. Peano continua which contain no circle are called *trees* by Menger [**10**], *dendrites* by Kuratowski and the Polish school. The term 'dendrite' now prevails. We need to consider noncompact analogues, so the term *tree* will be used to include those. The relation between compact and noncompact trees—separable trees, anyway—happens to be the simplest possible.

2.0. *Every separable tree T is dense in a unique compact tree, unique up to homeomorphism fixing T pointwise. Every tree has a unique smallest dense subtree.*

The result is betweenness-theoretic, i.e. it extends to suitable nonseparable spaces constructed out of connected linearly ordered spaces; but let us make use of the great convenience of a standard path $I = [0, 1]$ in the separable case. For the *meaning* of 2.0, (1) only the definition of *tree* is needed; given that, everything in 2.0 is meaningful. However, (2) though the present trees are more general than Menger's, they are not the most general reasonable 'arboreal' spaces, even apart from nonseparable monsters or e.g. the nonlocally connected 'dendroids' of the Polish school [**9**]. There are simplicial complexes which are countable combinatorial trees and, in the CW-topology or in a natural metrizable topology, are not embeddable in any compact tree. We have to choose a certain topology determined by the betweenness. Which topology? It is substantially obvious from the fact that a compact tree has a basis of open sets with finite boundaries [**10**] (which is betweenness-theoretic; part of the nonseparable generalization is in [**3**]).

That is why the definitions now following give first a "tree" without topology, and then the "coarse topology" ('coarse' justified by 2.0.4). The theorem, 2.0, is for (separable) trees *with the coarse topology.*

N. B. Some nonseparable trees have tree compactifications and some (e.g. the long line) do not. We can (without difficulty) delay assuming separability till near the end of the discussion.

We define a (*bare*) *tree* as a set T with a ternary (nonstrict) betweenness relation such that (i) for $a \neq b$ in T the set $[a, b]$ of points between a and b is betweenness-isomorphic with $[0, 1]$, (ii) $[a, b]$ is betweenness-closed, and (iii) if $[a, b] \cap [b, c] = \{b\}$, then $b \in [a, c]$. (Then, note, whenever $b \in [a, c]$, (i) and (ii) give $[a, c] = [a, b] \cup [b, c]$.) By a *subtree* is meant a betweenness-closed subset. (One might call these 'convex' or 'connected' sets; but I want to stress that such a tree as $[0,1) \cup \{2\} \subset [0,2]$ is *not* a subtree.)

2.0.1. *If* $[a,b] \cap [b,c]$ *is not* $\{b\}$*, it is an interval* $[m,b]$*, and* m *is the only point belonging to more than one of* $[a,m]$*,* $[b,m]$*, and* $[c,m]$*.*

PROOF. $M = [a,b] \cap [b,c]$ is a subtree of $[a,b]$ containing $\{b\}$, properly. Can it be a half-open interval $(I,b]$? If so, then in $[b,c]$ it is a half-open interval $[b,n)$, for the other possibility (closed) $[b,m]$ is not isomorphic with $(I,b]$. Now we can close in from a to I and from c to n; $[I,b] \supset (I,b]$ and $[b,n] \supset [b,n)$, so $[I,b] \cap [b,n]$ is still M. If $[I,n] \cap [I,b]$ were just $\{I\}$, then (iii) would give (since $[a,b] = [b,a]$ by (i)) $[b,n] \supset [b,I]$, $I \in [b,n]$, a contradiction. Similarly $[I,n] \cap [b,n]$ could not (in the supposed case) be just $\{n\}$. So $[I,n]$ contains an interval $[I,x]$ of $[I,b]$, x strictly between I and b; and it contains an interval $[y,n]$ of $[b,n]$, y strictly between b and n. x and y are in the half-open interval M closed at the end b. If z is the one of x and y farthest from b in M (i.e. the one such that $\{x,y\} \subset [b,z]$), there is w in M still farther from b. Now $[I,n]$ contains $[I,z]$ in which w is an interior point (i.e. $w \in [I,z] - \{I,z\}$); but also $[I,n]$ contains $[z,n]$ in which w is an interior point. Thus $[I,n]$ is not betweenness-isomorphic with $[0,1]$, a contradiction.

The only remaining possibility is that $M \subset [a,b]$ is a closed interval $[m,b]$, and it is also the interval $[m,b]$ in $[c,b]$. Since $[a,b]$ is isomorphic with $[0,1]$, $[a,m] \cap [m,b] = \{m\}$; since $[b,c]$ is likewise, $[b,m] \cap [m,c] = \{m\}$. $[a,m] \cap [m,c] \subset [a,b] \cap [b,c] = [m,b]$; so it is contained in $[a,m] \cap [m,b]$, which is $\{m\}$, completing the proof.

Observe that with (iii) this determines $[a,b] = [a,m] \cup [m,b]$ and similarly $[a,c]$ and $[b,c]$. In particular, $[a,c] \subset [a,b] \cup [b,c]$. This means

2.0.2. *Star-shaped sets are subtrees.*

That is, if $o \in S$ and for any $x, y, y \in S$ and $x \in [o,y]$ imply $x \in S$, then S is a subtree—since $[y,z] \subset [y,o] \cup [o,z]$. Hence

2.0.3. *Any subset* S *of a tree* T *is partitioned into subtrees of* T. For the union of all subtrees of S which include a point o is a subtree, and these partition S.

These maximal subtrees will be called the *components of* S. A *basic set* of a tree is a component of the complement of a finite set. Observe:

A nonempty intersection of two basic sets is basic.

For if V and W are components of the complements of finite F and G, respectively, and $p \in V \cap W$, the component of p in $T - F - G$ is $V \cap W$. (An intersection of betweenness-closed sets is of course betweenness-closed.)

The *coarse topology* is the lattice of unions of basic sets; that is, these are the coarse-open sets.

2.0.4. *The coarse topology on a bare tree is the smallest topology that has a basis of subtrees and agrees with the closed-interval topology on closed intervals.*

PROOF. It is clearly such a topology. Any such topology is T_1; no point a is in the closure of another point b, since this can be checked in the interval $[a,b]$. Then for finite $F \subset T$ and a betweenness-component C of $T - F$, each point p of C has a subtree neighborhood N_p contained in $T - F$; so $N_p \subset C$, and C is open. Thus all basic sets, and hence all coarse-open sets, are open.

Note: the coarse topology itself, and therefore all (these) finer ones, are Hausdorff. For two points p, r lie in distinct components of $T - \{q\}$, for any q in $[p,r]$.

The coarse topology is regular; a basic set containing a point p, which is its component in the complement of $\{r_1, \dots, r_n\}$, contains a closed neighborhood, viz. the closure of the component of p in the complement of $\{q_1, \dots, q_n\}$ where $q_i \in (p, r_i)$.

An *end point* of a tree is a point that is not between two others. We can now prove the second sentence of 2.0 in a few lines; and uniqueness in the first sentence, in a few more, relying on a theorem of Skljarenko [**13**]. Constructing the end-point (tree) compactification will take longer, but many readers may prefer their own arguments (or faith, the mover of mountains).

The smallest dense subtree of a tree (*in the coarse topology, or in any topology with a subtree basis, right on closed intervals*) *is the set* D *of all nonend points, except in a trivial tree* $\{o\}$.

PROOF. D is a subtree; in fact, every point between two other points is in D. (Equivalently, every subset of the tree T containing D is a subtree.) It is dense in every interval, hence dense in T since no point is isolated in each interval that includes it (trivial trees excepted). If E is a dense subtree of T (in one of these topologies, hence in the coarse topology by 2.0.4), consider any $q \in D$ and (p, r) with q in it. E must meet both the component of p and the component of r in $T - \{q\}$; say it meets them respectively in o, s. Neither $[o, p]$ nor $[r, s]$ includes q, so $[o, s]$ must include q—otherwise p and r would be in the same component of $T - \{q\}$. Thus D is contained in every dense subtree.

As for uniqueness (in the sense given in 2.0) of the tree compactification T^- of a (separable) tree T, this is an instance of uniqueness of the Freudenthal compactification, which Skljarenko proved [**13**] for a compactification constructed by adjoining a *nowhere cutting and punctiform* set. The set adjoined (if T^- is a tree and T a dense subtree) is a set of end points, and the set of all end points is obviously zero-dimensional, so punctiform. "Nowhere cutting" means that no point p of T^- has a basis of neighborhoods U with $U \cap T$ the sum of two relatively open sets both approaching p (as limit point), which is clear.

To existence. Since every subset containing D is a subtree, the tree compactification can be built up one point at a time. The stages will be:

2.0.5. *A noncompact tree has an end ray*;

2.0.6. *A separable tree having an end ray has a simple extension*;

and finally, a colimit construction (essentially Zorn's lemma) of the extension which cannot be extended, so must be compact. Separability will not be a problem; extensions of separable trees are separable. We need some definitions, as follows.

An *extension* of a tree T is a tree T' in which T is a dense subtree. It is a *simple extension* if T' has just one point not in T. A *chain* is a subset of a tree, of any three of whose points one is between the other two. An *end ray* is a chain with exactly one end point b which is neither properly contained in another chain of which b is an end point, nor equal to $\{b\}$.

PROOF OF 2.0.5. Let $\mathfrak{G}$ be a free closed ultrafilter (that is, a maximal proper filter of closed sets whose total intersection is empty). Fix a point o of T, and let R_o be the set of all points $p \neq o$ for which the component of o in $T - \{p\}$ is disjoint from some member of $\mathfrak{G}$, together with o. I claim that R_o is an end ray. First, it is a chain with o at one end. For (1) o is certainly not between two points p, q of R_o; this would mean that $\mathfrak{G}$ has elements A, B disjoint respectively from the sets of points x for which $p \notin [x, o]$ and for which $q \notin [x, o]$, but the union of those sets is T, so the intersection $A \cap B$ would be empty. Then, in the same way, (2) for p, q in R_o either $p \in [o,q]$ or $q \in [o,p]$ (since otherwise no $[x,o]$ contains both p and q). Applying (2) to p, q, r in R_o, one is between the other two. (This is analogous to 2.0.2; a set is a chain if it looks like a chain from one of its points.)

Further, R_o cannot have another end point e. Suppose it did. e is certainly not an end point of T—then the component of o in $T - \{e\}$ would be $T - \{e\}$, $\{e\} \in \mathfrak{G}$, so the intersection of $\mathfrak{G}$ would not be empty. So e is in some open interval (a,b), and therefore in an open interval (o,b) (or (o,a)). But still, $\mathfrak{G}$ has empty intersection, so it includes the complement of some basic set which includes e; there is finite $F = \{f_1, \ldots, f_n\} \subset T - \{e\}$ such that the set C of points x for which $[x,e]$ meets F is an element of $\mathfrak{G}$. Since C is the union of the closed sets C_i of points x for which $[x,e]$ includes f_i, some C_i belongs to $\mathfrak{G}$. Consider hows f_i lies relative to $[o,e]$; if $e \notin [o,f_i]$, put $b' = b$, but if $e \in [o,f_i]$ (thus in (o,f_i)), choose b' in (e,f_i). Now b' must be in R_o. For a point x in the component of o in $T - \{b'\}$ is either in the component of o in $T - \{e\}$ (the complement of a member of $\mathfrak{G}$), or $e \in [o,x]$ and $f_i \notin [e,x]$, which says $x \notin C_i$. So $b' \in R_o$, and e is after all not an end point.

A trivial argument (an inclusion) shows that R_o is star-shaped from o and thus a subtree. Then it is a maximal chain with o as an end point. If not, we would have $[o,e']$ properly containing R_o, and since there is only one end point, $R_o = [o,e)$ for some e, and a basic neighborhood of e with complement belonging to $\mathfrak{G}$ gives, as above, a contradiction. This proves 2.0.5.

Turning to separable trees, in the coarse topology, what we want is not the countable dense subset but the countability of disjoint collections of intervals, which is equivalent. Also, since our announced subject is Peano continua, we really need almost all of

2.0.7. *For a tree T, the following are equivalent*:

(a) *T is separable in the coarse topology.*

(b) *T is separable in every topology which agrees with the closed-interval topology on closed intervals.*

(c) *T contains no uncountable disjoint collection of open intervals.*

(d) *T is metrizable in the coarse topology.*

PROOF. (b) $\Rightarrow$ (a) trivially. Also, (a) $\Rightarrow$ (d). For if T has a countable dense set S in the coarse topology, the smallest subtree R containing S is a countable union of open intervals, and contains a countable set Q dense in every interval of it. Since R is a dense subtree, it contains D, and Q is dense in *every* interval. Now evidently the components of complements of finite subsets of Q form a basis. There is not an uncountable disjoint collection of open sets in a separable space, so that is a countable basis; T is second countable. We noted after 2.0.4 that T is regular, so (d) holds.

For (c) $\Rightarrow$ (b), the topology that matters is the finest one that is right on intervals: a set is open [closed] if this is true of its trace on every closed interval. Let C_0 be a maximal chain. Inductively, form $C_{\alpha+1}$ from C_α by adjoining a chain, maximal subject to having an end point b in C_α, but having no other point in C_α, and at a limit ordinal λ let C_λ be the closure (intervalwise) of the union of the preceding C_β. Inductively, each C_α is a closed subtree. It is easy to check that as long as $C_\alpha \neq T$ there is a larger $C_{\alpha+1}$. But each $C_{\alpha+1} - C_\alpha$ contains an open interval. Since these are disjoint, $C_\alpha = T$ for some countable ordinal α. Each chain

$C_{\alpha+1} - C_\alpha$ (satisfying (c)) has a countable dense set, and the union is a countable dense set in T.

For (d) $\Rightarrow$ (c), suppose (c) is false. How? If T contains a nonseparable chain, it contains a long line and is not metrizable. So in case every chain is separable, build up subtrees C_α as above; but now there must be uncountably many of them. Then some C_α contains end points of chains $C_{\beta+1} - C_\beta$ for uncountably many β. If (Case 1) one point is the end point of uncountably many such chains, the first axiom of countability fails there (in the coarse topology—for different $C_{\beta+1} - C_\beta$ are disjoint), so again (d) fails. In the remaining case (Case 2) the separable metric space C_α has an uncountable set S of such end points. Removing the set of points of C_α at which S is locally countable, almost all of S remains (all but a countable subset), and there remains a closed subspace H at each point of which $S \cap H$ is locally uncountable. Then it is easy to check that no σ-locally finite family of coarse-open sets contains neighborhood bases at uncountably many points of $S \cap H$. In no case is T metrizable, and the proof is complete.

PROOF OF 2.0.6. Let R be an end ray of separable T with end point o. Since there is no other end point, and (maximal) R is clearly a subtree, it is betweenness-isomorphic with $[0,1)$; add a point p to correspond to 1, and define $[r,p] = \{p\} \cup R - [o,r)$ for each $r \in R$. For $t \notin R$, $[o,t] \cap R$ is a subtree, $\neq R$ since that would contradict maximality of R. For any r in $R - [o,t]$, every s in $R \cap [o,t]$ is in $[o,r]$, since in the chain R with end point o the only other possibility is $r \in [o,s]$ which would put r in $[o,t]$. So $[r,o] \cap [o,t]$, an interval $[m,o]$ by 2.0.1, contains $R \cap [o,t]$. Also $[t,m] \cap [m,r] = \{m\}$, so $[t,r] = [t,m] \cup [m,r]$. But now with that fixed m, for any r' in $R - [o,t]$, $[r',o] \cap [o,t]$ is still exactly $[m,o]$, and $[t,r'] = [t,m] \cup [m,r']$. Accordingly define $[t,p]$ to be $[t,m] \cup R - [o,m)$. This completes the definition of the simple extension $T' = T \cup \{p\}$. By construction T is a dense subtree, and by definition T' satisfies axioms (i), (ii) for a tree. There is no trouble in checking axiom (iii), which completes the proof.

PROOF OF 2.0. Observe that a tree having a separable dense subtree is separable, so (in view of 2.0.7) has at most the power of the continuum, so that a transfinite process of building extensions must terminate. But the process is obvious; having the initial separable tree T, dense in extensions T_β for $\beta < \alpha$, with embeddings $T_\gamma \to T_\beta$ for $\gamma < \beta < \alpha$, take the colimit C_α—which is T_δ if $\alpha = \delta + 1$. In any case C_α is plainly a tree extending T. If it is compact, stop; if not, 2.0.5–6 give a simple extension T_α [with the help of the axiom of choice], so we continue until it does stop, T_α being a compact extension tree.

2.1. *Every Peano continuum X is an irreducible quotient of a tree T. If X has no open set homeomorphic with the real line, T may be chosen to have dense end points.*

[REMARK. To use the convex metrization theorem in proving this seems very like using some great and wonderful machine to crack a nut. But the nut does put up some resistance. Having a finite subtree T_n of X that is nearly dense, any one point of $X - T_n$ can be reached by adding a short whisker; but if finitely many points are added (to get twice as dense), how do we keep the whiskers from crossing without making long whiskers? The problem is essentially rectifiability. Use of the great theorem (Bing [**1**], Moise [**11**]) may be excessive, but it is not inappropriate.

Further: is the crossing problem imaginary? No. Consider a space composed of an H, made of 5 unit segments, and a long arrowhead $\triangleright$ attached at the right, with Euclidean distance. To the subtree T consisting of the long sides of the arrowhead, any one point can be added in a whisker of *diameter* $\sqrt{2}$ or less. But a larger tree containing the upper left and lower left points must reach one of them by a path from T of diameter $\sqrt{5}$. If we can use arc length instead of diameter, the problem vanishes.]

PROOF OF 2.1. Use a convex metric for X [**1**, **11**], of diameter 1. Let $P = \{p_i\}$ be a countable dense set in X. Then P is a union of initial segments P_n such that each point of X is within 2^{-n} of some point of P_n, beginning with $P_0 = \{p_1\}$. Build up spanning trees T_n for P_n as follows. Having T_k, take the points $p_r, p_{r+1}, \dots, p_s$ of $P_{k+1} - P_k$ in order; for each one, *consider* a shortest join $p_i x$ from it to some point of T_k, and *use* as much of it as is needed to join p_i to the spanning tree of $\{p_1, \dots, p_{i-1}\}$. The segments $p_i x$ are shortest joins of *each* of their points to some point of T_k; so the actual path in T_{k+1} from p_i to T_k is a shortest such path.

The union of all T_n is a separable tree T mapped one-to-one upon a dense subset of X. The mapping is continuous, coarse topology to topology of X, because this is true on the finite trees, and no path with just one end in T_n has length greater than 2^{1-n}. For the same reason, $T \to X$ extends to a continuous map of the tree compactification $\overline{T}$ onto X. Since P is dense in X, the construction secures that P is dense in $\overline{T}$, and mapped homeomorphically; so the mapping is irreducible.

How must this construction be modified to secure a tree $\overline{T}$ with dense end points? When the kth arc A is added to T_k, choose a basis $\{U_{ki}\}$ for A and, along a previously selected subsequence of the steps (say, those divisible by 2^k but not by 2^{k+1}), add neat little whiskers W_{ki} rooted somewhere in U_{ki}; this is possible, since A is nowhere dense.

2.2. *A Peano continuum is homeomorphic with every Peano continuum of which it is an irreducible quotient if and only if it is a tree whose set of end points has zero-dimensional closure and which has no cut point of infinite order.*

PROOF. '*If*'. Let $f\colon X \to T$ be irreducible, X Peano, T a suitable tree. We shall show that the set S of points p of T for which $f^{-1}(p)$ is a singleton is all of T. Note, S contains each open set V on which f has a continuous section $g\colon V \to X$, i.e. $fg(v) = v$ for all v in V. For $g(V)$ is relatively closed in $f^{-1}(V)$, being the set of fixed points of gf on $f^{-1}(V)$; so $f^{-1}(V) - g(V)$ is absolutely open, and its complement maps onto T.

Next, S includes all points having a neighborhood homeomorphic with the real line. For if p has such a neighborhood, consider a closed neighborhood $[a, b]$ contained in it, and $x \in f^{-1}(\{a\})$, $y \in f^{-1}(\{b\})$, and a simple arc xy. There is a subarc C mapping onto $[a, b]$—take the last point of xy with image "$\le a$", that is, in the same component of $T - \{p\}$ as a and not in (a, p), and the first point thereafter with image "$\ge b$". By construction $f|C$ maps only one point to a and one to b; since $f^{-1}((a, b))$ is open, $f|C$ is irreducible and therefore (exercise) invertible.

Next,

(*) *S contains every open set W whose intersection with $T - S$ is zero-dimensional.*

Each $p \in W$ has a closed basic neighborhood $N \subset W$. Now $f^{-1}(p)$ is completely arcwise connected [**8**], i.e. each simple arc in X joining two points x, y of $f^{-1}(p)$ lies entirely in $f^{-1}(p)$. If not, since $f^{-1}(p)$ is closed, there would be a subarc A

having just its end points x', y' in $f^{-1}(p)$. Then $A - \{x', y'\}$ would map into one component C of $T - \{p\}$, whose closure has p as an end point. But small relative neighborhoods of x' and y' map to arcs in C ending at p, therefore meeting in a continuum in C, and in S; A is not a simple arc.

But this implies that each component of $X - f^{-1}(p)$ has a singleton boundary [**8**, 47.1.4]. There may be more components than $T - \{p\}$ has. However, if two components K_1, K_2 of $X - f^{-1}(p)$ map into the same component C of $T - \{p\}$, they have the *same* singleton boundary. For, the boundary point q_i of K_i is arcwise accessible from K_i. Arcs A_i in K_i^- ending at q_i map to arcs in C^- ending at p, which meet in a continuum in C, and thus in S: a contradiction. So (in our trees) $f^{-1}(p)$ has finite boundary. But by irreducibility it is nowhere dense, and it is completely arcwise connected; that is, it is a singleton. So $W \subset S$, which is $(*)$.

But in view of $(*)$, S contains $T - E^-$. For points not in E^- have basic neighborhoods with closures K disjoint from E. Then (each) K is a compact tree with finitely many end points, thus a finite tree, with the given point in K^o, an open set which is a 1-manifold except at a zero-dimensional set. And now $(*)$ completes the proof of 'if'.

'*Only if*'. By 2.1, these Peano continua must be trees. If a tree T has a cut point p of infinite order (necessarily countable), p can be blown up to an arc A with the components of $T - \{p\}$ attached (one each) at a dense set of points of A. If the closure of the set of end points is not zero-dimensional it contains an arc A; $T - A$ is easily seen to have infinitely many components, attached at a dense set of points of A, and A can be blown up to a 2-cell.

The continua of 2.2 may be called *isolated d-initial.* The proof shows that not only are their predecessors under 'irreducible quotient' homeomorphic, but the only irreducible maps are homeomorphisms. Concerning merely *d-initial* Peano continua K (that is, if $C \to K$ is irreducible there is also an irreducible surjection $K \to C$) I know nearly nothing. There are some nonisolated ones, e.g. any nondegenerate Peano continuum C with one whisker attached at each point of a countable dense set in C. Irreducible maps between these correspond to continuous maps $C \to C'$ taking the dense set in C bijectively to the dense set in C', which always exist [**2**]. On the other hand, the proof of 2.2 shows that an irreducible map f from a Peano continuum X to a whiskered C must be invertible on the whiskers. $f^{-1}(C)$ must be completely arcwise connected as in 2.2 (indeed, by quoting the proof of 2.2 applied to the result of pinching C to a point), and this means X is just a whiskered C'.

What about nonlocally connected continua? The metric continua mapping irreducibly to the Hilbert cube, or to its irreducible quotients, are precisely the nondegenerate ones [**2**]. Mapping irreducibly to I, the basic nonlocally connected example is the closure of the graph of $y = \sin(1/x)$, projecting to $x \in I$. Of course one can have a dense set of "limit lines", i.e. nonsingleton fibers. It must be first category since there is a dense G_δ set of singletons. (Actually, the union of the singleton fibers is exactly a G_δ.) It is easy to see that the metric continua mapping irreducibly to I are locally connected at each singleton-fiber point. The fibers are connected, but otherwise unrestricted (individually, that is).

The *d-final* Peano continua (apart from a point) are just the irreducible quotients of a Hilbert cube H, since H is an irreducible quotient of each nondegenerate Peano continuum [**2**]. They are described by

2.3. *Every nondegenerate Peano tree with dense end points is d-final. Thus this is true of every nondegenerate Peano continuum which has no open subset homeomorphic with R.*

PROOF. First we describe an irreducible map $f\colon I^2 \to T$, for any tree T with dense end points—equivalently, with branch points dense in every interval. (An open interval without branch points would be a coarse-open set containing no end point.) Choose an end point b_0 of T, which will be the image of the 1-sphere B_0 bounding I^2. There are only countably many branch points (if not, some interval would contain uncountably many, giving rise to uncountably many disjoint intervals); list them, $b_1, b_2, \ldots$. We shall associate to each branch point b_i, of order n_i (≥ 3), a set B_i $(= f^{-1}(b_i))$ which is topologically a circle with $n_i - 2$ circles inside it but tangent to it, at different points—and of diameters $\to 0$ if n_i is infinite.

Where is B_i? There are two types of conditions: (1) it is properly placed vis-a-vis the preceding B_j, and (2) as $i \to \infty$, they are so arranged as to squeeze each end (except b_0) to diameter 0. It is convenient to associate not only B_j to b_j, but the n_j components of $I^2 - B_j$ to the n_j components of $T - \{b_j\}$. With this, clearly (1) can be taken care of; components of $T - \{b_j\}$, for various $j < i$, which meet must correspond to components of $I^2 - B_j$ which meet, B_i goes in the right intersection, et cetera. As for (2), when constructing B_i, whichever of its constituent circles surround no previous B_j are to be of diameter at most $1/i$. (When all previous b_j lie in the same component of $T - \{b_i\}$, this will mean that B_i has such a diameter; otherwise only some of the inner circles are affected.) Then in travelling in T toward any end point $(\neq b_0)$, through the dense branch points, infinitely often one will enter a component corresponding to a component in I^2 of diameter at most $1/i_k$, and one ends in a set of diameter 0. Completing the verification for I^2 is a bit beside the point, as we need to adapt the construction to a Hilbert cube.

Evidently the same thing can be done in I^m with $(m-1)$-spheres. One cannot just pass to the limit; in fact a Hilbert cube H does not have a closed boundary B_0. The simplest procedure seems to be to treat H as inverse limit of I^m. Do the first step, selecting B_1, in I^2—a circle not touching the boundary, with $n_1 - 2$ circles tangent to it inside. More precisely, such a figure in I^2 is A_1, say, and B_1 is its inverse image in H. Force the component-diameter condition (this time) by taking H of diameter 1. Then for each B_i, add a dimension. Thus in I^3 the trace of B_1 is a bunch of cylinders tangent along generators. All of it will be outside A_2, and A_2 must be so chosen that B_2 will have diameter at most $1/2$. Then this can all be carried out. It gives us a mapping f_0 defined on $\bigcup B_n$, dense in H, to $\{b_n\}$, dense in T. f_0 is uniformly continuous. Indeed, for suitable initial segments S of $\{b_1, b_2, \ldots\}$, the diameter of the union of two neighboring components of $T - S$ is $< \varepsilon$, whence sufficiently nearby points in $\bigcup B_n$ have images distant $< \varepsilon$. So there is a unique continuous extension $f\colon H \to T$. The inverse image of every point is located by the components of complements of B_n's in which it lies. In particular, the inverse image of an end point is a singleton, so f is irreducible, as was to be shown.

Note, (1) the stages in which we have mapped the Hilbert cube H to any sufficiently complicated Peano continuum Y give the monotone-light factorization; the map to a tree, $H \to T$, is monotone, and $T \to Y$ is light. (2) The attractive result 1.5 is, sad to say, unnecessary for these maps; inspection shows a particular dense G_δ set on which the map of 2.1 is an embedding, and another (somewhat less visible; the inverse image of the endpoints) for the map of 2.3.

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