AN EMPTY CLASS OF NONMETRIC SPACES

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ABSTRACT. Let CSSM be the class of compact nonmetrizable spaces in which every subspace of cardinality at most ω₁ is metrizable. We show that CSSM is empty.

For the purposes of this article only let us call a space an SSM space (small subspaces metrizable) if it is not metrizable but it is regular and all of its subspaces of cardinality at most ω₁ are metrizable. A CSSM space is a compact SSM space. If X is a CSSM space, then X is first countable [HJ]. Therefore under the continuum hypothesis (CH) there are no CSSM spaces because, of course, a compact first countable space has cardinality at most c. This was first observed by Juhasz who then asked if the CH assumption could be removed [J]. It was shown in [D] that it is consistent with (and independent of) ¬CH that there are no Lindelöf, countably compact or even ω₁-compact first countable SSM spaces. In this article we show that there simply are never any CSSM spaces.

There is however an easy example, under MA + ¬CH, of a Lindelöf first countable SSM space. I do not know if a Lindelöf SSM space is necessarily first countable.

EXAMPLE 1. Recall that the Alexandroff double topology on I × 2 (where I is the unit interval) is obtained by declaring I × {1} to be open and discrete while a basic open neighbourhood of a point (r, 0) is U × 2 − {(r, 1)} where r ∈ U is open in I. If A ⊆ I is any uncountable set containing no uncountable closed set, then X = (I − A × {0}) ∪ (A × {1}) is a Lindelöf non metrizable subspace of the Alexandroff double.

Furthermore, if MA(ω₁) is assumed then X is an SSM space since A × {1} will be an Fₙ-set in any subspace of X of cardinality ω₁ (see [M]).

One might hope to modify the Alexandroff double somehow to obtain a CSSM space. In fact if X were a CSSM space then X would contain an uncountable discrete subset D; hence cl D would itself be a CSSM space. (To see that X would contain such a D see 2(ii).)

2. PROPOSITION. If X is a SSM space then:
   (i) each separable subspace is metrizable and
   (ii) X contains an uncountable discrete subset and
   (iii) if X is, in addition, compact then X contains an uncountable discrete set D whose closure is a CSSM space.

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(i) is essentially due to Hajnal and Juhasz [HJ]. They prove that a space has countable weight if all of its subspaces of cardinality at most \( \omega_1 \) do. Now (i) follows since if \( K \subseteq X \) is separable, each subspace of size \( \omega_1 \) is contained in a separable metrizable space (since \( X \) is SSM and \( K \) is separable).

Of course (ii) is obvious since, by (i), \( X \) is not separable and therefore \( X \) has a non separable subspace of cardinality \( \omega_1 \). This subspace is metrizable, hence not ccc. For (iii), let \( D \) be given by (ii) and note that \( \text{cl} \, D \) is not metrizable and is therefore a CSSM space.

3. **CSSM is empty.** Suppose that \( X \) is a compact space containing the discrete space \( \omega_1 \) as a dense subspace. For each \( x \in X \) fix a countable neighbourhood base \( \{U(x, n) : n < \omega\} \). Let \( \mathcal{U} = \{\{U(x, n) : n < \omega\} : x \in X\} \).

For each \( \lambda < \omega_1 \) let \( E \lambda = \bigcap\{\text{cl}(\lambda - \alpha) : \alpha < \lambda\} \).

We shall define by induction on \( \gamma < \omega_1 \) a continuous increasing sequence \( \{M(\gamma) : \gamma < \omega_1\} \) of countable elementary submodels of some sufficiently large \( H(\theta) \) and for each \( \lambda(\gamma) = M(\gamma) \cap \omega_1 \) we will choose \( x(\gamma) \) in \( E \lambda(\gamma) \) as follows.

Suppose \( \gamma < \omega_1 \) and \( \{M(\rho) : \rho < \gamma\} \), \( \{x(\rho) : \rho < \gamma\} \) have been chosen so that \( \{X, \mathcal{U}\} \subseteq M(\rho) \) and \( \{x(\rho), M(\rho)\} \subseteq M(\rho + 1) \) for each \( \rho < \gamma \).

In case \( \gamma \) is a limit, let \( M(\gamma) = \bigcup\{M(\rho) : \rho < \gamma\} \). If \( \gamma = \rho + 1 \) let \( M(\gamma) \) be any countable elementary submodel of \( H(\theta) \) containing \( \{M(\rho), x(\rho)\} \).

The following fact is probably of some interest by itself and it provides the basis for the whole proof.

**FACT 1.** If \( M \) is a countable elementary submodel of \( H(\theta) \) such that \( X, \mathcal{U} \) are in \( M \), \( \lambda = M \cap \omega_1 \), and if \( F \in [X \cap M]^{\omega} \) and \( p : F \to \omega \) are such that \( E \lambda \subseteq \bigcup\{U(x, p(x)) : x \in F\} \) then

\[
[\beta, \omega_1) \subset \bigcup\{U(x, p(x)) : x \in F\}.
\]

**PROOF.** By definition of \( E \lambda \), each sequence cofinal in \( \lambda \) is eventually in \( \bigcup\{U(x, p(x)) : x \in F\} \), hence there is some \( \beta < \lambda \) such that \( [\beta, \lambda) \subseteq \bigcup\{U(x, p(x)) : x \in F\} \). But since \( F, p \) and \( \omega_1 \) are all in \( M \), we have that \( M \) is a model of \( [\beta, \omega_1) \subseteq \bigcup\{U(x, p(x)) : x \in F\} \). Now the fact follows since \( M \) is an elementary submodel.

**NOTATION.** For \( p < \gamma \) let \( U(p, n) = U(x(p), n) \).

**FACT 2.** There is an \( x(\gamma) \in E \lambda(\gamma) \) such that for any \( p < \gamma \) and \( n < \omega \), \( x(\gamma) \subseteq U(p, n) \rightarrow \lambda(\gamma) \subseteq U(p, n) \).

**PROOF.** If not we could find for each \( x \in E \lambda(\gamma) \) a pair \( t(x) \in \gamma \times \omega \) such that \( x \subseteq U(t(x)) \) and \( \lambda(\gamma) \not\subseteq U(t(x)) \). Since \( E \lambda(\gamma) \) is compact we find \( F \in [\gamma]^{<\omega} \) such that \( \bigcup\{U(t(x)) : x \in F\} \supseteq E \lambda(\gamma) \). However, this contradicts Fact 1 since \( \lambda(\gamma) \not\subseteq U(t(x)) \) for \( x \in F \).

Therefore we have defined a cub \( \{\lambda(\gamma) : \gamma < \omega_1\} \), a sequence \( \{x(\gamma) : \gamma < \omega_1\} \) with \( x(\gamma) \in E \lambda(\gamma) \) and a sequence of neighbourhood bases \( \{U(\gamma, n) : \gamma < \omega_1, n < \omega\} \) so that \( p < \gamma \) and \( x(\gamma) \subseteq U(p, n) \rightarrow \lambda(\gamma) \subseteq U(p, n) \).

**FACT 3.** \( \omega_1 \cup \{x(\gamma) : \gamma < \omega_1\} \) is not metrizable.

**PROOF.** Assume that it is metrizable. Recall that each open subset of a metric space is an \( F_\sigma \). Therefore there must be a stationary set \( S \subseteq \{\lambda(\gamma) : \gamma < \omega_1\} \) such that \( \text{cl} \, S \cap \{x(\gamma) : \gamma < \omega_1\} \subseteq \emptyset \) since \( \omega_1 \) is open. For each \( \lambda(\gamma) \in S \) choose \( n(\gamma) < \omega \) so that \( U(x(\gamma), n(\gamma)) \cap S = \emptyset \). It follows that if \( \lambda(\gamma) < \lambda(p) \) are both in \( S \) then \( x(p) \not\subseteq U(x(\gamma), n(\gamma)) \) since \( \lambda(p) \not\subseteq U(x(\gamma), n(\gamma)) \). Let, for \( n \in \omega \), \( \mathcal{V}_n \) be a locally finite family of open subsets of \( \omega_1 \cup \{x(\gamma) : \gamma < \omega_1\} \) such that \( \bigcup\mathcal{V}_n : n \in \omega \) is a
base (recall that each metric space has a \(\sigma\)-locally finite base). For each \(\lambda(\gamma) \in S\), there is an \(m_\gamma\) and a \(V_\gamma \in \mathcal{V}_{m_\gamma}\) such that \(x(\gamma) \in V_\gamma \subseteq U(x(\gamma), n(\gamma))\). There is an \(m \in \omega\) such that \(S' = \{\lambda(\gamma) \in S : m_\gamma = m\}\) is stationary. Since \(\lambda(\gamma) < \lambda(\rho)\) both in \(S'\) implies \(x(\rho) \notin V_\gamma\), these sets are all distinct (i.e. \(V_\gamma \neq V_\rho\)). However, for each \(\lambda(\gamma) \in S'\), \(V_\gamma \cap [0, \lambda(\gamma)) \neq \emptyset\) since \(x(\gamma)\) is a limit point of \([0, \lambda(\gamma))\). Now a pressing down argument gives that the family \(\{V_\gamma : \lambda(\gamma) \in S'\}\) is not point-finite, contradicting that \(\mathcal{V}_m\) is locally finite.

**FACT 4.** \(X\) is not SSM.

**REFERENCES**


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