RELATION BETWEEN GROWTH AND REGULARITY OF SOLUTIONS OF HYPOELLIPTIC EQUATIONS

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(Communicated by Walter D. Littman)

ABSTRACT. For a class of linear partial differential equations with variable coefficients, it is shown that the Gevrey regularity of solutions depends on their growth at infinity.

Let \( P(D) \) be a partial differential operator with constant coefficients. If \( P(D) \) is hypoelliptic, then all distributions \( u \) in \( \mathbb{R}^n \), solutions of the equation

\[
P(D)u = 0,
\]

are \( C^\infty \)-functions which belong to a Gevrey class \( \Gamma^d(\mathbb{R}^n) \), where \( d = (d_1, \ldots, d_n) \) and \( d_j \geq 1, j = 1, \ldots, n \). This means that for every compact set \( K \subset \mathbb{R}^n \) there is a constant \( C > 0 \) such that

\[
|D^\alpha u(x)| \leq C|\alpha| + 1 \alpha_1 d_1 \alpha_2 d_2 \cdots \alpha_n d_n, \quad x \in K
\]

for every multi-index \( \alpha \).

V. V. Grusin [2] has shown that, for a given solution \( u \) of equation (1), the Gevrey class \( \Gamma^d(\mathbb{R}^n) \) depends not only on the differential operator \( P(D) \) but also on the growth of \( u \) at infinity. In fact, the numbers \( d_j, j = 1, \ldots, n \), can be lowered and condition (2) can be replaced by a global condition in \( \mathbb{R}^n \), if one considers solutions of finite exponential order of growth in \( \mathbb{R}^n \).

The aim of this paper is to extend Grusin’s investigations to a class of partial differential operators with variable coefficients.

1. The case of operators with constant coefficients. We recall briefly some of the results obtained in [2].

Let \( P(D) \) be a differential operator with constant coefficients and \( P(\xi) \) the corresponding polynomial. We denote by \( \mathcal{N}_k \) the set of all \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n \) such that \( P(\xi) = 0 \) and \( \text{Im} \xi_j = 0 \) for \( j \neq k \). If \( P(D) \) is hypoelliptic, then there are rational numbers \( d_j^k > 0 \) and constants \( C_j^k > 0, j, k = 1, \ldots, n \), with the following properties:

\( (h_1) \) If \( \xi = \xi + i\eta \), where \( \xi = (\xi_1, \ldots, \xi_n) \) and \( \eta = (\eta_1, \ldots, \eta_n) \) are in \( \mathbb{R}^n \), then

\[
|\xi_j| \leq C_j^k (1 + |\eta|) d_j^k = C_j^k (1 + |\eta_k|) d_j^k.
\]

\( (h_2) \) \( d_j^k \) are the smallest numbers for which the inequalities (3) are valid with some constants \( C_j^k \).
We call the numbers \( d_j^k \) the exponents of hypoellipticity of the operator \( P(D) \). We note that \( d_j^k \) corresponds to \( 1/\gamma_j^k \) in [2].

It is well known that every differential operator \( P(D) \) with constant coefficients has a tempered fundamental solution, i.e. there exists a tempered distribution \( E \) such that

\[
(4) \quad P(D)E = \delta
\]

where \( \delta \) is the Dirac measure. For a hypoelliptic operator \( P(D) \), every tempered fundamental solution \( E \) has the following properties (see [2, Theorem 3.1]):

1. If \( d_j^k, k, j = 1, \ldots, n, \) are exponents of hypoellipticity of \( P(D) \) and \( d_j = \max_{1 \leq k \leq n} d_j^k \geq 1 \), then \( E \in \Gamma^d(R^n \setminus \{0\}) \), where \( d = (d_1, \ldots, d_n) \).

2. There exists an integer \( l \geq 0 \) such that

\[
(5) \quad D^\alpha E(x) = O((1 + |x|)^l) \quad \text{as} \quad |x| \to \infty,
\]

for every multi-index \( \alpha \).

In particular, the property (e1) implies that every distribution \( u \), solution of equation (1) in \( R^n \), belongs to \( \Gamma^d(R^n) \) (see [4, Proposition 7.2, p. 413]).

We now state a version of the theorem of Grusin [2, Theorem 4.1]) which establishes the relation between growth and regularity of solutions of equation (1).

**THEOREM 1.** Let \( P(D) \) be a hypoelliptic differential operator and \( d_j^k, j, k = 1, \ldots, n, \) its exponents of hypoellipticity. Furthermore, let \( u \) be a solution of equation (1) which satisfies the growth condition

\[
(6) \quad |u(x)| \leq A \exp \left( a \sum_{k=1}^{n} |x_k|^{p_k} \right), \quad x \in R^n,
\]

where \( A \) and \( a \) are positive constants and \( p_k > 1, k = 1, \ldots, n, \). Then there exist constants \( C > 0 \) and \( c \geq a \) such that

\[
(7) \quad \left| \frac{\partial^m u(x)}{\partial x_j^m} \right| \leq AC^m \left( \sum_{k=1}^{n} m^{md_j^k/q_k} \right) \exp \left( c \sum_{k=1}^{n} |x_k|^{p_k} \right), \quad x \in R^n,
\]

where \( m = 1, 2, \ldots, \) and \( 1/p_k + 1/q_k = 1 \).

**REMARK.** Since \( p_k > 1, \) clearly \( d_j^k/q_k < d_j^k, j, k = 1, \ldots, n, \) which shows that the regularity is improved due to the restriction of the growth of \( u \).

2. The case of operators with variable coefficients. We consider a differential operator of the form

\[
(8) \quad P(x, D) = P_0(D) + \sum_{\nu=1}^{r} a_\nu(x)P_\nu(D)
\]

where \( P_\nu(D) \) are operators with constant coefficients and the functions \( a_\nu \) satisfy certain regularity and growth conditions. Specifically, we make the following assumptions on the operators \( P_\nu(D) \):

1. The operator \( P_0(D) \) is hypoelliptic. We denote by \( d_j^k, j, k = 1, \ldots, n, \) its exponents of hypoellipticity and assume that \( d_j^k > 1 \).
(c2) For \( j = 1, \ldots, n \) and \( \nu = 1, \ldots, r \),
\[
\int_{\mathbb{R}^n} \frac{\hat{P}_\nu(\xi)}{P_0(\xi)} \, d\xi < \infty,
\]
where \( \hat{P}(\xi) = (\sum_\alpha |P^{(\alpha)}(\xi)|^2)^{1/2} \) and \( P^{(\alpha)}(\xi) = D^\alpha P(\xi) \).

Note that, because of (c1), \( P_0(D) \) satisfies the conditions imposed on \( P(D) \) in Theorem 1. Also, condition (c2) implies that, for any \( x_0 \in \mathbb{R}^n \), \( P(x_0, D) \) and \( P_0(D) \) are equally strong. Hence \( P(x_0, D) \) is hypoelliptic for every \( x_0 \in \mathbb{R}^n \), by (c1) and Theorem 4.1.6 in [3]. Moreover, if \( a_\nu, \nu = 1, \ldots, r \), are \( C^\infty \)-functions then every distribution \( u \) in \( \mathbb{R}^n \), solution of the equation
\[
P(x, D)u = 0,
\]
is a \( C^\infty \)-function, by Theorem 7.4.1 in [3]. We wish to study solutions of equation (10) which satisfy the growth condition (6).

We make the following assumption on the functions \( a_\nu \):
(c3) Let \( \rho_j = \min_{1 \leq k \leq n} d_j^k/q_k \), \( j = 1, \ldots, n \), and let \( c \) be the constant in (7), when Theorem 1 is applied to the operator \( P_0(D) \). Then there exist constants \( B > 0 \) and \( b > c \) such that
\[
|D^\alpha a_\nu(x)| \leq B^{|\alpha|+1} a_1^{\alpha_1} \cdots a_n^{\alpha_n} \rho_n \exp \left(-b \sum_{k=1}^n |x_k|^{p_k}\right), \quad x \in \mathbb{R}^n
\]
for all \( \nu = 1, \ldots, r \), and all multi-indices \( \alpha \).

REMARK. Since \( d_j^k < 1 \) and \( 1/p_k + 1/q_k = 1 \), we have \( 1/p_k + d_j^k/q_k > 1, j, k = 1, \ldots, n \), and therefore the family of functions satisfying condition (c3) is not trivial, i.e. it contains functions that are not identically zero (see [1, Chapter IV, §8]).

Our main result is the following theorem.

**THEOREM 2.** Let \( P(x, D) \) be a differential operator of the form (8), where the operators \( P_0(D) \) and \( P_\nu(D) \) satisfy conditions (c1) and (c2), and the functions \( a_\nu \) satisfy condition (c3). If \( u \) is a solution of equation (10) which satisfies the growth condition (6), then there are positive constants \( A_1, C_1 \) and \( c_1 > a \) such that
\[
\left| \frac{\partial^m u(x)}{\partial x_j^m} \right| \leq A_1 C_1^m \left( \sum_{k=1}^n m^{m(1+d_j^k/q_k)} \right) \exp \left(c_1 \sum_{k=1}^n |x_k|^{p_k}\right), \quad x \in \mathbb{R}^n
\]
where \( m = 1, 2, \ldots, \) and \( 1/p_k + 1/q_k = 1 \).

We first prove two lemmas. In the first lemma, \( \mathcal{S} \) denotes Schwartz’s space of rapidly decreasing \( C^\infty \)-functions.

**LEMMA 1.** Let \( P(D) \) and \( Q(D) \) be differential operators with constant coefficients and suppose that \( P(D) \) is hypoelliptic. If
\[
\int_{\mathbb{R}^n} \frac{\hat{Q}(\xi)}{\hat{P}(\xi)} \, d\xi < \infty,
\]
and if \( E \) is a fundamental solution for \( P(D) \), then there are constants \( M \) and \( N \) such that
\[
|(Q(D)E * \phi)(x)| \leq M(1 + |x|)^N \int_{\mathbb{R}^n} (1 + |y|)^N |\phi(y)| \, dy, \quad x \in \mathbb{R}^n,
\]
where “*” denotes the convolution and \( \phi \in \mathcal{S} \).
PROOF. Since $P(D)$ is hypoelliptic, there are positive constants $a, c$ and $C$ such that

$$|\xi|^c \leq C|P(\xi)| \quad \text{for } \xi \in G_a = \{\eta \in \mathbb{R}^n; |\eta| \geq a\}.$$ 

Consider the function

$$\Phi(\xi) = \begin{cases} \frac{1}{P(-\xi)} & \text{for } \xi \in G_a, \\ 0 & \text{otherwise}. \end{cases}$$

If $\hat{E}$ is the Fourier transform of $E$, then $P\hat{E} = 1$. In particular, $\hat{E}(\xi) = 1/P(\xi)$ for $\xi \in G_a$. It follows that $H = (\hat{E})^* - \Phi$ is a distribution with compact support and we have

$$E = \hat{H} + \Phi; \quad (\hat{E})^* \text{is defined by } (\hat{E})^*(\phi) = \hat{E}(\phi), \quad \text{where } \hat{\phi}(\xi) = \phi(-\xi).$$

By the Paley-Wiener-Schwartz theorem, $\hat{H}$ is an entire function of exponential type and there are positive constants $M_1$ and $N$ such that

$$|Q(D)\hat{H}(x)| \leq M_1(1 + |x|)^N, \quad x \in \mathbb{R}^n.$$ 

We note that $N$ does not depend on the operator $Q(D)$. It follows that, for any $\phi \in \mathcal{S}$, we have

$$|(Q(D)\hat{H} * \phi)(x)| \leq M_1 \int_{\mathbb{R}^n} (1 + |x - y|)^N |\phi(y)| \, dy \leq M_1(1 + |x|)^N \int_{\mathbb{R}^n} (1 + |y|)^N |\phi(y)| \, dy, \quad x \in \mathbb{R}^n. \quad (16)$$

Also,

$$\sqrt{Q(D)\hat{H} * \phi} = Q(\xi)\hat{\Phi}(\xi)\hat{\phi}(\xi),$$

so that

$$(Q(D)\hat{H} * \phi)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x,\xi)} Q(\xi)\hat{\Phi}(\xi)\hat{\phi}(\xi) \, d\xi.$$ 

Hence

$$|(Q(D)\hat{H} * \phi)(x)| \leq \frac{1}{(2\pi)^n} \int_{G_a} \frac{|Q(\xi)|}{P(\xi)} |\hat{\phi}(\xi)| \, d\xi, \quad x \in \mathbb{R}^n.$$ 

Using again the hypoellipticity of $P(D)$, we can find a constant $C' > 0$ such that

$$\tilde{P}(\xi) \leq C'|P(\xi)|, \quad \xi \in G_a.$$ 

Consequently,

$$|(Q(D)\hat{H} * \phi)(x)| \leq \frac{C'}{(2\pi)^n} \sup_{\xi \in \mathbb{R}^n} |\hat{\phi}(\xi)| \int_{G_a} \frac{|Q(\xi)|}{\tilde{P}(\xi)} \, d\xi \leq M_2 \int_{\mathbb{R}^n} |\phi(y)| \, dy, \quad x \in \mathbb{R}^n, \quad (17)$$

where

$$M_2 = \frac{C'}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\tilde{Q}(\xi)}{\tilde{P}(\xi)} \, d\xi.$$ 

If $M = M_1 + M_2$, we obtain from (15), (16) and (17) the estimate (14).
COROLLARY. If \( P(D), Q(D) \) and \( E \) are as in Lemma 1 and

\[
\int_{\mathbb{R}^n} \frac{\xi_j Q(\xi)}{P(\xi)} d\xi < \infty,
\]

for some \( j \), then

\[
\left| (Q(D)E \ast \frac{\partial \phi}{\partial x_j}) (x) \right| \leq M (1 + |x|)^N \int_{\mathbb{R}^n} (1 + |y|)^N |\phi(y)| dy, \quad x \in \mathbb{R}^n.
\]

For the proof it suffices to apply Lemma 1 to the operator \( D_j Q(D) \), where \( D_j = \partial / \partial x_j \).

REMARK. Lemma 1 remains valid, if we assume that \( \phi \) is a continuous function rapidly decreasing at infinity, i.e. that \( (1 + |x|)^k \phi (x) \) is bounded in \( \mathbb{R}^n \), for every \( k \).

LEMMA 2. Let \( P(x, D) \) be a differential operator of the form (8), where \( P_0(D), P_\nu(D) \) and \( a_\nu, \nu = 1, \ldots, r \), satisfy conditions (c1), (c2) and (c3), and let \( E_0 \) be a tempered fundamental solution for \( P_0(D) \). Then there are operators \( Q_\nu(D) \) and functions \( b_\nu, \nu = 1, \ldots, s \), with the following property. If \( u \) is a solution of the equation (10) which satisfies condition (6), then

\[
u = u + \sum_{\nu=1}^s Q_\nu(D)E_0 \ast (b_\nu u)
\]
is a solution of the equation

\[
P_0(D) \nu = 0.
\]

Each polynomial \( Q_\nu(\xi) \) is a derivative of some order of a polynomial \( P_\mu(\xi), \mu \geq 1 \), and each function \( b_\nu(\xi) \) is proportional to a derivative of some order of a function \( a_\mu(\xi) \).

PROOF. The polynomials \( Q_\nu(\xi) \) and the functions \( b_\nu(\xi) \) obviously satisfy the conditions (9) and (11), respectively. In particular, the products \( b_\nu u \) decrease at infinity faster than any power of \( |\xi|^{-1} \). Therefore the convolutions in (19) are well defined.

By assumption

\[
P_0(D)u + \sum_{\nu=1}^r a_\nu P_\nu(D)u = P(x, D)u = 0.
\]

To each term \( a_\nu P_\nu(D)u \) we now apply the generalized Leibniz formula

\[
a_\nu P_\nu(D)u = \sum_{\alpha} \frac{(-1)^{|\alpha|}}{\alpha!} P_\nu^{(\alpha)}(D)(uD^\alpha a_\nu).
\]

In this way we obtain the equation

\[
P_0(D)u + \sum_{\nu=1}^s Q_\nu(D)(b_\nu u) = 0.
\]

Since \( E_0 \) is a fundamental solution for \( P_0(D) \), we have

\[
Q_\nu(D)(b_\nu u) = P_0(D)[Q_\nu(D)E_0 \ast (b_\nu u)]
\]
for each \( \nu = 1, \ldots, s \). Hence

\[
P_0(D) \left[ u + \sum_{\nu=1}^{s} Q_\nu(D) E_0 \ast (b_\nu u) \right] = 0
\]

which proves the lemma.

**Proof of Theorem 2.** If \( u \) satisfies condition (6), then \( v \) satisfies the same condition with another constant \( A_0 \geq A \). Since, by Lemma 2, \( v \) is a solution of equation (20), we may apply Theorem 1 to conclude that

\[
u = - \sum_{\nu=1}^{s} Q_\nu(D) E_0 \ast (b_\nu u) + f,
\]

where \( Q_\nu(D) \) and \( b_\nu \) are as in Lemma 2 and \( f \) is a \( C^\infty \)-function which satisfies the estimates (7). In particular, condition (18) is valid for \( j = 1, \ldots, n \), with \( Q(\xi) \) replaced with \( Q_\nu(\xi) \), \( \nu = 1, \ldots, s \), and the functions \( b_\nu \) satisfy condition (11) with the constants \( B \) and \( b \). Thus, in view of the corollary from Lemma 1, we have

\[
\frac{\partial u(x)}{\partial x_j} \leq M \sum_{\nu=1}^{s} (1 + |x|)^N \int_{\mathbb{R}^n} (1 + |y|)^N |b_\nu(y)u(y)| \, dy + \left| \frac{\partial f(x)}{\partial x_j} \right|
\]

\[
\leq A_0 C_1 (1 + \lambda s M) \exp \left( c \sum_{k=1}^{n} |x_k|^{p_k} \right), \quad x \in \mathbb{R}^n
\]

where \( C_1 = \max\{B, C\} \) and

\[
\lambda = \sup_{x \in \mathbb{R}^n} \left[ (1 + |x|)^N \exp \left( -c \sum_{k=1}^{n} |x_k|^{p_k} \right) \right] \cdot \int_{\mathbb{R}^n} (1 + |y|)^N \exp \left[ -(b - c) \sum_{k=1}^{n} |x_k|^{p_k} \right] \, dy.
\]

Suppose now that

\[
\left| \frac{\partial^l u(x)}{\partial x_j^l} \right| \leq A_0 C_1^l (l + \lambda s M)^l \left( \sum_{k=1}^{n} t^{d_k/q_k} \right) \exp \left( c \sum_{k=1}^{n} |x_k|^{p_k} \right), \quad x \in \mathbb{R}^n
\]

for \( l = 1, \ldots, m \). Then

\[
\left| \frac{\partial^{m+1} u(x)}{\partial x_j^{m+1}} \right| \leq M \sum_{\nu=1}^{s} (1 + |x|)^N \int_{\mathbb{R}^n} (1 + |y|)^N \left| \frac{\partial^m [b_\nu(y)u(y)]}{\partial y_j^m} \right| \, dy + \left| \frac{\partial^{m+1} f(x)}{\partial x_j^{m+1}} \right|
\]

\[
\leq A_0 C_1^{m+1} \left\{ \lambda s M \sum_{\nu=0}^{m} \binom{m}{l} t^{\rho_j} (l + \lambda s M)^{m-l} \left( \sum_{k=1}^{n} (m-l)^{(m-l)d_k/q_k} \right) \right. \]

\[
\left. + \left( \sum_{k=1}^{n} (m+1)^{(m+1)d_k/q_k} \right) \right\} \cdot \exp \left( c \sum_{k=1}^{n} |x_k|^{p_k} \right), \quad x \in \mathbb{R}^n.
\]
Since, by definition, \( \rho_j = \min_{1 \leq k \leq n} d_j^k / q_k \), we have

\[
|\partial^{m+1} u(x)| \leq A_0 C_1^m (m + 1 + \lambda sM)^m + 1 \left( \sum_{k=1}^{n} (m + 1) d_j^k / q_k \right) 
\cdot \exp \left( c \sum_{k=1}^{n} |x_k| p_k \right) , \quad x \in \mathbb{R}^n.
\]

This proves, by induction, that for any \( j = 1, \ldots, n \), and \( m = 1, 2, \ldots \),

\[
|\partial^m u(x)| \leq A_0 C_1^m (m + \lambda sM)^m \left( \sum_{k=1}^{n} m d_j^k / q_k \right) \exp \left( c \sum_{k=1}^{n} |x_k| p_k \right) 
\leq A_0 e^{\lambda sM} C_1^m \left( \sum_{k=1}^{n} m (1 + d_j^k / q_k) \right) \exp \left( c \sum_{k=1}^{n} |x_k| p_k \right) , \quad x \in \mathbb{R}^n.
\]

The theorem is thus established with the constants

\[
A_1 = A_0 e^{\lambda sM}, \quad C_1 \quad \text{and} \quad c_1 = c.
\]

**Remark.** If \( 1 < \max_{1 \leq k \leq n} d_j^k / p_k \), then \( 1 + \max_{1 \leq k \leq n} d_j^k / q_k < \max_{1 \leq k \leq n} d_j^k \), i.e. the regularity of the solution \( u \) in the \( j \)th variable is improved due to the growth condition (6).
REFERENCES


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