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SINGULAR FOLIATION C^*-ALGEBRAS

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ABSTRACT. A way to construct C^*-algebras from the so-called singular foliations is proposed, which generalizes Connes' construction of C^*-algebras of regular foliations. A certain desired property is shown to hold for such C^*-algebras and some connection with the boundary behavior of pseudodifferential operators is indicated.

Introduction. In recent years, geometers have shown a lot of interest in the generalization of foliation, called singular foliation, i.e. a manifold foliated by submanifolds of various dimensions in a nice way, and obtained fundamental results about it [**Daz**, **Eec**, **Ehr**]. On the other hand, A. Connes has successfully established a relation between regular foliations and operator algebras by constructing foliation algebras [**Con**] from the geometry of foliations and opened a new area of mathematics. It is of course natural to ask how one can extend the construction to include singular foliations. This question is quite interesting because it has been known for a long time that group C^*-algebras of Lie groups have close connection with singular foliations on the coadjoint spaces through the orbit method. On the other hand this question is also not obvious since a straight generalization does not work because the holonomy groupoid of a singular foliation is usually bad and one cannot form the groupoid C^*-algebra of it [**Ren**].

The following work results from the author's first attempt to construct C^*-algebras from singular foliations in a reasonable way to reflect some geometry of the foliations. Although the result is not enough to justify this construction to be "the" right generalization, it is hoped that this topic and some interesting questions arising from this work will catch people's attention.

1. In this section we shall review the definition and properties of the so-called singular foliations studied by Pierre Dazord [**Daz**], Paul ver Eecke [**Eec**] and Charles Ehresmann [**Ehr**].

Let $\mathscr{F}$ be a collection of disjoint connected submanifolds F, called leaves, of a manifold M such that $M = \bigcup_{F\in\mathscr{F}} F$ and, for any x in M, let F_x be the unique F in $\mathscr{F}$ containing x. Then $(M,\mathscr{F})$ is called a singular foliation [**Daz**] if for any x in M and any v in $T_x(F_x)$, the tangent space to F_x at x, there is a C^∞ vector field χ on M such that $\chi(x) = v$ and $\chi(y) \in T_y(F_y)$ for all y in M.

Now we slightly generalize the above notion of singular foliation to include (foliated) varieties which are not manifolds. (In fact, the following generalization turns out to be included in the more general setting studied by C. Ehresmann [**Ehr**].) Let

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V be a Hausdorff space with a partition $\mathscr{F}$ (i.e. a collection of disjoint connected subsets of V) such that for each x in V, there are a neighborhood U of x and a finite union M of submanifolds M_i's of $\mathbf{R}^n$, where M is also a union of leaves of some singular foliation $(\mathbf{R}^n, \mathscr{F}_0)$, such that U can be identified with M through a homeomorphism in the way that $\mathscr{F}|_U$ (i.e. the collection of connected components of $F \cap U, F \in \mathscr{F}$) coincides with $\mathscr{F}_0$ restricted to M. Such a homeomorphism ϕ from U into $(\mathbf{R}^n, \mathscr{F}_0)$ is called a distinguished chart map. Furthermore, we assume the smooth compatibility between distinguished charts. More precisely, the chart transformation $\phi' \circ \phi^{-1}$ between overlapping charts U and U' when restricted to each $\phi^{-1}(U \cap U') \cap M_i$ is a diffeomorphism onto $\phi'(U \cap U') \cap M'_j$ for some M'_j in M'. From now on, we shall call such a pair $(V, \mathscr{F})$ a singular foliation and the subsets F in $\mathscr{F}$ leaves.

Clearly, Dazord's result [**Daz**] on the local product structure of singular foliations holds in the generalized case, namely, for any x in V there are a neighborhood U of x and a singular foliation $(V', \mathscr{F}')$ containing a singleton leaf $\{p\}$ such that $(U, \mathscr{F}|_U)$ can be identified with $(V' \times \mathbf{R}^m, \mathscr{F}' \times \mathbf{R}^m)$ as singular foliations for some $m \in \mathbf{N}$ so that x is identified with $(p, 0)$ where $\mathscr{F}' \times \mathbf{R}^m$ is the collection of the products $F \times \mathbf{R}^m$, $F \in \mathscr{F}'$. (Note that Weinstein's splitting theorem for Poisson structure [**Wein**] is an interesting special case of Dazord's result.)

We can either use the theory of the holonomy groupoid established in [**Ehr**, **Eec**] for locally simple topological foliations or use the above structure theorem to construct directly the holonomy groupoid $\mathfrak{G}$ of a singular foliation $(V, \mathscr{F})$ as one usually does for regular foliations. (To avoid technical difficulty, we only consider singular foliations which are locally simple.)

2. Let $(V, \mathscr{F})$ be a singular foliation and $\mathfrak{G}$ be its holonomy groupoid. Let V_m be the union of leaves of dimension m. Then on V_m we have a smooth (positive) half line bundle $|\bigwedge^m \mathscr{F}|$, the bundle of one-densities along the leaves in V_n. By fixing a smooth section μ_m of $|\bigwedge^m \mathscr{F}|$, the union of the half line bundles $|\bigwedge^m \mathscr{F}|$ becomes a trivial half line bundle $|\bigwedge \mathscr{F}|$ over V. Note that in general the groupoid $\mathfrak{G}$ is not locally compact and it has no nice Haar system [**Ren**] which one usually uses to construct C^*-algebras from groupoids. Let s and t be the source and target maps defined on $\mathfrak{G}$. For any $p \in \mathfrak{G}$ we can find neighborhoods U and W of $s(p)$ and $t(p)$ respectively, such that p is a one-to-one correspondence between the leaf spaces $U/U \cap \mathscr{F}$ and $W/W \cap \mathscr{F}$. We define $G(U, p, W)$ to be the subset $\{(x, y) | x \in U, y \in W \text{ and } p([x]) = [y]\}$ of $U \times W$. Then such $G(U, p, W)$'s form a fundamental system of open sets of $\mathfrak{G}$.

Now we construct the foliation algebra of $\mathscr{F}$ in the following way. Let

$$G(U, p, W)_m = G(U, p, W) \cap s^{-1}(V_m).$$

Define $\Gamma(U, p, W)$ to be the set of bounded complex valued functions ξ on $G(U, p, W)$ satisfying the following conditions: (1) ξ is continuous on each $G(U, p, W)_m$; (2) the closure of $\operatorname{supp}(\xi)$ in $G(U, p, W)$ is compact; (3) under convolution with respect to the fixed measure μ, ξ and ξ^* preserve continuous functions on V, i.e. $\xi * f \in C(V)$ for all $f \in C(V)$ and $\xi^* * g \in C(V)$ for $g \in C(V)$ where $\xi^*(x, y) := \overline{\xi(y, x)}$: (4) $\|\xi\|_1 < \infty$, where

$$\|\xi\|_1 = \max\left\{\int^x |\xi(x, y)|\mu(x), \int^y |\xi(x, y)|\mu(y)\right\}_{x \in U, y \in W}.$$

Let $\mathscr{A}(\mathfrak{G},\mu)$ be the linear span of the elements in such $\Gamma(U,p,W)$'s. It is routine to check that $\mathscr{A}(\mathfrak{G},\mu)$ is an involutive algebra normed by

$$\|\xi\|_1 := \max(\int_{t(p)=y} |\xi(p)|(x^s(\mu))(p), \int_{s(p)=x} |\xi(p)|(t^*(\mu))(p)\}.$$

For each leaf F, the left convolution (w.r.t. μ) gives rise to a regular representation π_F of $\mathscr{A}(\mathfrak{G},\mu)$ on the Hilbert space $L^2(F,\mu)$ of μ-square integrable functions on F. Let $\xi \in \Gamma(U,p,W)$, $f \in L^2(F,\mu)$ and $g \in L^2(F,\mu)$. Then

$$\begin{aligned}
\int^y |(\xi * f)g|\mu &\le \int_W^y \int_U^x |\xi(x,y)f(x)g(y)|\mu(x)\mu(y) \\
&\le \left(\int_W^y \int_U^x |\sqrt{|\xi(x,y)|}f(x)|^2\mu(x)\mu(y)\right)^{1/2} \\
&\quad \times \left(\int_W^y \int_U^x |\sqrt{|\xi(x,y)|}g(y)|^2\mu(x)\mu(y)\right)^{1/2} \\
&\le \left(\int_U^x \left(\int_W^y |\xi(x,y)|\mu(y)\right)|f(x)|^2\mu(x)\right)^{1/2} \\
&\quad \times \left(\int_W^y \left(\int_U^x |\xi(x,y)\mu(x)|\right)|g(y)|^2\mu(y)\right)^{1/2} \\
&\le \|\xi\|_1^{1/2}\|f\|_2\|\xi\|_1^{1/2}\|g\|_2 = \|\xi\|_1\|f\|_2\|g\|_2.
\end{aligned}$$

So we get $\|\xi * f\|_2 \le \|\xi\|_1\|f\|_2$, hence $\|\pi_F(\xi)\| \le \|\xi\|_1$. (Actually, this can be shown to be true for all ξ in $\mathscr{A}(\mathfrak{G},\mu)$ since $s^{-1}(x)$ and $t^{-1}(x)$ are covering spaces of F_x [**Eec**] and $\overline{\mathrm{supp}(\xi)}^{\mathfrak{G}}$ is compact.) The algebra $\mathscr{A}(\mathfrak{G},\mu)$ normed by $\|\xi\| = \sup_{F\in\mathscr{F}}\|\pi_F(\xi)\|$ is a pre-C^*-algebra, whose completion $C^*(V,\mathscr{F},\mu)$ is called a (reduced) foliation C^*-algebra of $(V,\mathscr{F})$.

When $\mathfrak{G}$ is Hausdorff, we can define $C^*(V,\mathscr{F},\mu)$ in a global way. Indeed, we consider the involutive algebra $\mathscr{B}(\mathfrak{G},\mu)$ of complex valued functions ξ on $\mathfrak{G}$, satisfying the following properties: (1) ξ is a bounded compactly supported (in $\mathfrak{G}$) function continuous on each $\mathfrak{G}_m = s^{-1}(V_m)$, (2) ξ and ξ^* preserve $C(\mathfrak{G})$ under convolution w.r.t. μ, (3) $\|\xi\|_1 < \infty$. Then $\mathscr{B}(\mathfrak{G},\mu)$ is an involutive algebra normed by $\|\ \|_1$, which clearly contains $\mathscr{A}(\mathfrak{G},\mu)$. When $\mathfrak{G}$ is Hausdorff, it is routine to check that $\mathscr{A}(\mathfrak{G},\mu) = \mathscr{B}(\mathfrak{G},\mu)$ by using a partition of unity.

When $\mathscr{F}$ is a regular foliation, it is clear that $C^*(V,\mathscr{F})$ is the foliation C^*-algebra introduced and studied by A. Connes [**Con**].

3. The study of C^*-algebras associated with singular foliations was motivated by group C^*-algebras of Lie groups which are closely related to the singular foliations gotten from the Poisson manifold structure on the coadjoint spaces [**Wein**, **Kir**]. In [**Sheu**], an induction process was based on the fact that in the singular foliation associated with group G certain embedded singular foliations are associated with quotient groups of G, whose group C^*-algebras are quotients of $C^*(G)$. Thus it is natural to ask whether $C^*(V',\mathscr{F}',\mu)$ is a quotient of $C^*(V,\mathscr{F},\mu)$ when $(V',\mathscr{F}')$ is a closed subfoliation in $(V,\mathscr{F})$. With some reasonable assumption on μ, the answer is likely affirmative. We shall prove this for the following simple situation.

Let M be an n-dimensional manifold with boundary ∂M foliated by $\mathscr{F} = \{M\backslash\partial M\} \cup \{\{x\}|x \in \partial M\}$ and μ be a measure on M which is smooth on $M\backslash\partial M$ and the point mass at each $x \in \partial M$. It is plausible that if μ blows up at each boundary point in the sense that every neighborhood of a boundary point has infinite measure then $C^*(\partial M, \mathscr{F}|\partial M, \mu|\partial M) = C_0(\partial M)$ is a quotient of $C^*(M, \mathscr{F}, \mu)$. In fact, for any η in $C_0(\partial M)$, one can easily construct a function ξ on $\mathfrak{G}$ which restricts to η on ∂M satisfying all the properties (1)—(4) in §2 except that ξ^* also preserves continuous functions, by suitably controlling the y-supports of $\xi(x, \cdot)$'s. (Thus we can get a nonselfadjoint algebra with $C_0(M)$ as a quotient in this way.) In order to have ξ^* preserving $C_0(\partial M)$, we need to have control of the overlappings of the y-supports of $\xi(x, \cdot)$'s, which is technically more difficult. We can do it under a mild regularity of μ which is satisfied by most of the measures that people are interested in. The technical part is contained in the proof of the following lemma which is a special case of the next theorem with a stronger assumption on μ.

LEMMA. *Given M, ∂M, $\mathscr{F}$ and μ as above, the algebra $C^*(\partial M, \mathscr{F}|\partial M, \mu|\partial M) = C_0(\partial M)$ is a quotient of $C^*(M, \mathscr{F}, \mu)$ if for each $p_0 \in \partial M$ there are a neighborhood U of p_0 and a chart $\phi\colon U \to [0,1) \times \mathbf{R}^{n-1}$ with $\phi(p_0) = 0$ and $\phi(U \cap \partial M) \subseteq \{0\} \times \mathbf{R}^{n-1}$ such that $\phi_*(\mu) = w(x)dx$ and $w(x) = O(x_1^r)$ (i.e. for some $0 < a < b < \infty$, $ax_1^r < |w(x)| < bx_1^r$ for x_1 close to 0) with $r < -1$ for $x \in (0,1) \times \mathbf{R}^{n-1}$.*

PROOF. It turns out easier to work with a local chart which is an unbounded domain with boundary at the infinity than with a bounded one. So we first make a change of variables. Let p_0, ϕ, w and r be as in the statement. Define $\psi\colon [0,1) \times \mathbf{R}^{n-1} \to (1,\infty] \times \mathbf{R}^{n-1}$ by $\psi(x_1, x_2, \ldots, x_n) = (x_1^{-1/s}, x_2, \ldots, x_n)$, $s > 0$, where $0^{-1/s} = \infty$ is understood. Now $\psi_*(\phi_*(\mu)) = \psi_*(w(x)dx) = sw(\psi^{-1}(x))x_1^{-1-s}dx = O(x_1^{-1-(r+1)s})dx$. Let $v(x) = sw(\psi^{-1}(x))x_1^{-1-s}$ and $\alpha = -1-(r+1)s$. Note that since $r < -1$, we can find $s > 0$ such that $0 < \alpha < n/(n-1)$ and hence $(1/\alpha) - 1 > -1/n$. Now since $(d/dt)(t^{1/\alpha}) = (1/\alpha)t^{(1/\alpha)-1} > (1/\alpha)t^{-1/n}$ for $t > 1$, there is a constant $c > 0$ such that if $v(x) = t$ is sufficiently large then $t + 1 > v(y) > t - 1$ for $|x - y| < c(t-2)^{-1/n}$.

In constructing a ξ with desired boundary behavior by specifying the y-supports of $\xi(x, \cdot)$ (the following $B(x)$'s), we need to make sure that they overlap in a reasonable way in order to have nice x-supports of $\xi(\cdot, y)$ so that ξ^* also has the desired boundary behavior. Thus we need the following estimate in which the growth condition of μ specified in the lemma is used to ensure that the size of $B(x)$ is sufficiently large but not too large.

For x_1 sufficiently large, there is a unique (open) ball $B(x)$ of radius $r(x_1)$ centered at x such that $((\psi \circ \phi)_*(\mu))(B(x)) = k_n c^n$ where k_n is the constant $\text{Volume}(B(x))/r(x_1)^n$. Since $v(y) < t+1$ for $|x-y| < c(t+1)^{-1/n} < c(t-2)^{-1/n}$, we have $r(x_1) \geq c(t+1)^{-1/n}$. Similarly, since $v(y) > t-1$ for $|x-y| < c(t-1)^{-1/n}$, we have $r(x_1) \leq c(t-1)^{-1/n}$.

Let $E = \{(p,q)|p \in U\backslash\partial M, q \in (\psi \circ \phi)^{-1}(B(\psi(\phi(p))))\}$. Clearly $\int^q \chi_E(p,q)\mu(q) = k_n c^n$ for all $p \in U\backslash\partial M$ close to p_0. We claim that $\int^p \chi_E(p,q)\mu(p)$ converges to $k_n c^n$ as q converges to points in $U \cap \partial M$ close to p_0. In fact, for $|x-y| < c(v(y)+2)^{-1/n}$, we have $r(x_1) \geq c(t+1)^{-1/n} > c(v(y)+2)^{-1/n}$ and hence $y \in B(x)$, since $v(y) + 1 > t = v(x)$. On the other hand, if $y \in B(x)$ then $|x-y| < r(x_1) \leq c(t-1)^{-1/n}$, hence $t + 1 > v(y) > t - 1$, so $|x-y| < c(t-1)^{-1/n} < c(v(y)-2)^{-1/n}$. Thus the set $\{(\psi \circ \phi)(p)|(p,q) \in E\}$ for fixed q contains the ball of radius

$c(v(y)+2)^{-1/n}$ centered at $y = (\psi \circ \phi)(q)$ and is also contained in the ball of radius $c(v(y)-2)^{-1/n}$ centered at y. Now by the fact that $v(y)+1 > v(x) > v(y)-1$ for $|x-y| < c(v(y)-2)^{-1/n}$ we get

$$\begin{aligned}(v(y)+1)k_n c^n (v(y)-2)^{-1} &\geq \int^p \chi_E(p,q)\mu(p) \\ &\geq (v(y)-1)k_n c^n (v(y)+2)^{-1}\end{aligned}$$

where $y = \psi(\phi(q))$. Thus $\int^p \chi_E(p,q)\mu(p)$ approaches $k_n c^n$ as q approaches the boundary ∂M (and hence $v(y)$ goes to the infinity).

It is easy to check that E is an open set since $r(x_1)$ is continuous in x_1. Since $r(x_1)$ goes to 0 as x_1 goes to the infinity, the union of the diagonal of $(U \cap \partial M)^2$ and the closure of E in $(U \backslash \partial M)^2$ is locally compact. One can find a continuous bump function $\varsigma(p,q)$ (bounded by 1) supported in E such that both $\int^q (\chi_E - \varsigma)(p,q)\mu(q)$ and $\int^p (\chi_E - \varsigma)(p,q)\mu(p)$ approach 0 as p and q approach the boundary ∂M and hence both $\int^q \varsigma(p,q)\mu(q)$ and $\int^p \varsigma(p,q)\mu(p)$ approach $k_n c^n$ as p and q approach ∂M.

For any $\eta \in C_c(U)$ we define $\xi(p,q) = \eta(p)\varsigma(p,q)$ for $p,q \in U \backslash \partial M$ and $\xi(p,p) = k_n c^n \eta(p)$ for $p \in U \cap \partial M$. Then the support of ξ in $\Delta(U \cap \partial M)^2 \cup (U \backslash \partial M)^2$ is compact. For any $f \in C(U)$, by the continuity of η and f, we have $\int^p \eta(p)\varsigma(p,q)f(p)\mu(p) \to k_n c^n \eta(q')f(q')$ as $q \to q' \in \partial M$ and $\int^q \overline{\eta(p)\varsigma(p,q)}f(q)\mu(q) \to k_n c^n \overline{\eta(p')}f(p')$ as $p \to p' \in \partial M$. In other words, $\xi * f, \xi^* * f \in C(U)$ for any $f \in C(U)$. So $(k_n c^n)^{-1}\xi$ is an element of $\mathscr{A}(\mathfrak{G},\mu)$ whose restriction to $\partial M \cong \Delta(\partial M)^2$ is $\eta|_{\partial M}$. Note that $\pi\colon \mathscr{A}(\mathfrak{G},\mu) \to C_0(\partial M)$ sending ξ to $\xi|_{\partial M}$ can be extended to a continuous map from $C^*(M,\mathscr{F},\mu)$ to $C_0(\partial M)$. In fact, π is the direct integral of π_F's, $F = \{p\} \subseteq \partial M$. Now since for any $p_0 \in \partial M$ and any continuous η supported in some neighborhood of p_0 there is ξ in $\mathscr{A}(\mathfrak{G},\mu)$ such that $\pi(\xi) = \eta$, we get π surjective. Q.E.D.

It is not hard to check that if $r > -1$ in the lemma then the map π in the proof is trivial, i.e. $\xi|_{\partial M} = 0$ for all $\xi \in \mathscr{A}(\mathfrak{G},\mu)$.

THEOREM. *Let M be an n-dimensional manifold with boundary ∂M foliated by $\mathscr{F} = \{M \backslash \partial M\} \cup \{\{x\} | x \in \partial M\}$ and μ be a measure on M which is smooth on $M \backslash \partial M$ and the point mass at each $x \in \partial M$. Then $C^*(\partial M, \mathscr{F}|\partial M, \mu|\partial M) = C_0(\partial M)$ is a quotient of $C^*(M,\mathscr{F},\mu)$ if for each $p_0 \in \partial M$ there are a neighborhood U of p_0 and a chart $\phi\colon U \to [0,1) \times \mathbf{R}^{n-1}$ with $\phi(p_0) = 0$ and $\phi(U \cap \partial M) \subseteq \{0\} \times \mathbf{R}^{n-1}$ such that $\phi_*(\mu) = w(x)dx$ for some positive function w on $(0,1) \times \mathbf{R}^{n-1}$ with $\int_0^1 w(x)dx_1 = +\infty$ for all fixed $x_2, x_3, \ldots, x_n$.*

PROOF. By the above lemma, all we need to do is to prove that under a suitable change of variables, the weight function w becomes $O(x_1^r)$ with $r < -1$.

In fact, let $\psi(x)_1 = f_x^{-1}((r+1)^{-1}(1 - x_1^{r+1}))$ where $f_x(t) = \int_t^1 w(x)dx_1$ is a diffeomorphism from $(0,1)$ onto $(0,+\infty)$ and $r < -1$. Define $\psi(x)_i = x_i$ for $1 < i$. Then ψ extends to a diffeomorphism on $[0,1) \times \mathbf{R}^{n-1}$. Replacing the chart ϕ by

$\Phi = \psi^{-1} \circ \phi$, we get

$$\begin{aligned}\Phi_*(\mu) &= \psi^*(w(x)dx) = w(\psi(x))((\partial/\partial x_1)\psi(x)_1)\,dx \\ &= w(\psi(x))(f'_x(\psi(x)_1))^{-1}(-x_1^r)\,dx \\ &= w(\psi(x))(-w(\psi(x)))^{-1}(-x_1^r)\,dx = x_1^r\,dx. \quad \text{Q.E.D.}\end{aligned}$$

Let F be the leaf $M\backslash\partial M$, then the representation π_F restricted to $\ker(\pi)$ is faithful since $\pi_F \oplus \pi$ is a faithful representation of $C^*(M,\mathscr{F},\mu)$. It is easy to see that $\ker(\pi)$ contains $\mathscr{K}$, the compact operators on $L^2(M\backslash\partial M,\mu)$. In general $\ker(\pi)$ is larger than $\mathscr{K}$ and its structure is not clear. It is important to understand the structure of $\ker(\pi)/\mathscr{K}$ from the index theory point of view [**Upm**].

A similar example is the $x-y$ plane foliated by straight lines $x = c$, $c \neq 0$, and discrete points $(0,y)$, $y \in \mathbf{R}$. In this situation, $C_0(y\text{-axis})$ is also a quotient of $C^*(\mathbf{R}^2,\mathscr{F},\mu)$ where μ is $|c|^{-1}dy$ on $x = c \neq 0$ and unit mass at $(0,y)$. It can be shown that $C^*(\mathbf{R}^2,\mathscr{F},\mu)$ is a subalgebra of $C_b(\mathbf{R}\backslash\{0\}) \otimes \mathscr{K}$ and the group C^*-algebra $C^*(H)$ of the Heisenberg Lie group is a subalgebra of $C^*(\mathbf{R}^2,\mathscr{F},\mu)$.

The splitting theorem of Dazord mentioned in §1 is helpful in studying such a question by reducing a foliation algebra to the stabilization of a simpler foliation algebra since the trivial foliation has $\mathscr{K}$ as its algebra. For example, if we foliate a manifold M (with boundary) by $\mathscr{F} = \{M\backslash\partial M, \partial M\}$, then $\mathscr{F}$ restricted to a tublar neighborhood of ∂M is isomorphic to the product foliation $([0,1)\times\partial M, \mathscr{F}_0 \times \partial M)$ where $\mathscr{F}_0 = \{\{0\},(0,1)\}$. Then it is not hard to show that $\mathscr{K}(L^2(\partial M,\nu)) = C^*(\partial M,\{\partial M\},\nu)$ is a quotient of $C^*(M,\mathscr{F},\mu)$ if μ is a smooth measure on $M\backslash\partial M$ and ν is the measure on ∂M such that $\mu = (t^r dt)\times\nu$ on the tubular neighborhood $(0,1)\times\partial M$ with $r \leq -1$.

4. Singular foliation algebras are closely related to the boundary behavior of pseudodifferential operators on manifolds with boundary. For example, in the case considered in the above theorem, (if ξ is smooth) $\pi_F(\xi)$ is a pseudodifferential operator on $L^2(M\backslash\partial M,\mu)$ of order 0 with the principal symbol vanishing since $\xi(x,y)$ is a bounded proper kernel function (i.e. $\operatorname{supp}(\xi(K,\cdot))$ is relatively compact for any compact K in $M\backslash\partial M$), but $\pi_F(\xi)$ is not compact if $\pi_{\partial M}(\xi) \neq 0$, where $F = M\backslash\partial M$. Locally, identifying U with $[0,1)\times\mathbf{R}^{n-1}$ through a chart ϕ, we have, for a smooth ξ in $\mathscr{A}$ with support in $U\times U$ and for x in $U\backslash\partial M$,

$$\int \xi(x,y)\mu(y)u(y)dy = \int e^{ix\cdot z}p_x(z)\hat{u}(z)dz =: (p(x,D)u)(x)$$

where $\check{p}_x(y) = \xi(x,x-y)\mu(x-y)$. Thus

$$\begin{aligned}p_x(z) &= \int e^{-iy\cdot z}\xi(x,x-y)\mu(x-y)dy \\ &= e^{-ix\cdot z}\int e^{iy\cdot z}\xi(x,y)\mu(y)dy\end{aligned}$$

which converges to $e^{-it\cdot z}(e^{it\cdot z}\xi(t,t)) = \xi(t,t)$ when x converges to $t \in \partial M$. Thus $\pi_{\partial M}$ can be regarded as some kind of boundary symbol map. Note that π_F is a faithful representation of $\mathscr{A}(\mathfrak{G},\mu)$ since $\pi_F(\xi)$ is the integral operator with kernel function ξ, but it is not necessarily faithful on $C^*(M,\mathscr{F},\mu)$ since the "boundary symbol map" sending $\pi_F(\xi) \in \mathscr{B}(L^2(M\backslash\partial M,\mu))$ to $\pi_{\partial M}(\xi) \in C_0(\partial M)$ may not be continuous w.r.t. the operator norm of $\pi_F(\xi)$.

It is interesting to note that, when M is foliated by $\{M\backslash\partial M, \partial M\}$ with μ as specified at the end of §3, we have compact operator valued boundary symbol map defined on $\mathscr{A}(\mathfrak{G}, \mu)$ (cf. [**Bou**]).

REFERENCES

[**Bou**] L. Boutet de Monvel, *Boundary problems for pseudo-differential operators*, Acta Math. **126** (1971), 11–51.

[**Con**] A. Connes, *A survey of foliations and operator algebras*, Proc. Sympos. Pure Math., vol. 38, part I, Amer. Math. Soc., Providence, R.I., 1982, pp. 521–528.

[**Daz**] P. Dazord, *Holonomie des feuilletages singuliers*, C. R. Acad. Sci. Paris Sér. 1 **298** (1984), 27–30.

[**Eec**] P. ver Eecke, *Sur le groupe fondamental d'un feuilletages singulier*, C. R. Acad. Sci. Paris Sér. 1 **301** (1985), 837–840.

[**Ehr**] C. Ehresmann, *Structures feuilletées*, Proc. 5th Canad. Math. Congress, 1961, pp. 109–172.

[**Kir**] A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Russian Math. Surveys **17** (1962), 53–104.

[**Ren**] J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Math., vol. 793, Springer-Verlag, Berlin and New York, 1980.

[**Sheu**] A. J. L. Sheu, *A cancellation theorem for modules over the group C^*-algebras of certain nilpotent Lie groups*, Canad. J. Math. **39** (1987), 365–427.

[**Wein**] A. Weinstein, *The local structure of Poisson manifolds*, J. Differential Geom. **18** (1983), 523–557.

[**Upm**] H. Upmeier, *Index theory for multivariable Wiener-Hopf operators*, J. Reine. Angew. Math. **384** (1988), 57–79.

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