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THE LEFSCHETZ NUMBER OF SELF-MAPS OF LIE GROUPS

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ABSTRACT. In this note we present a simple approach to the Lefschetz number for the self-maps of Lie groups. As an application it is proved that for any map $f\colon G \to G$ of a compact connected Lie group G, there is a solution to $(f(x))^k = x$ for some $k \le \lambda + 1$, where λ is the rank of the group G.

Let G be an n-dimensional compact connected Lie group with multiplication μ, inverse T and unit e. Let $[G, G]$ be the set of homotopy classes of maps $G \to G$. Given two maps $f, f'\colon G \to G$, we write $f \cdot f'$ to denote the map $G \to G$ defined by

$$(f \cdot f')(g) = \mu(f(g), f'(g)), \qquad g \in G.$$

All cohomology in this paper will be over a coefficient field F of zero characteristic.

Let Lef: $[G, G] \to Z$ be the function that sends each element in $[G, G]$ to its Lefschetz number. Then the classical Lefschetz fixed point theorem states that "if $f\colon G \to G$ is a map with $\mathrm{Lef}(f) \neq 0$, then f has a fixed point." Now we define another function $B\colon [G, G] \to Z$ by setting $B(f) = \operatorname{degree} f \cdot T$. Since $B(f) \neq 0$ implies $e \in \operatorname{Im} f \cdot T$, this function also possesses the property that "if $B(f) \neq 0$, f has a fixed point." The following assertion was suggested by Professor Jiang Boju.

THEOREM 1. *The two functions* $\mathrm{Lef}, (-1)^n B\colon [G, G] \to Z$ *coincide.*

Before setting out to prove this result, we fix some notation and recall some facts about Lie groups. Given a point $g \in G$ and a differentiable map $F\colon G \to G$, write G_g to denote the tangent space to G at g and $d_g F$, the differential of F at g. Let $L_g, R_g\colon G \to G$ be respectively the left translation $L_g(g') = \mu(g, g')$, and the right translation $R_g(g') = \mu(g', g)$. Then there is a natural homomorphism Ad, the adjoint representation, from G to $\mathrm{GL}(G_e)$ (the nonsingular linear transformation group of G_e) defined as follows:

$$\mathrm{Ad}(g) = d_g R_{g^{-1}} \circ d_e L_g = d_{g^{-1}} L_g \circ d_e R_{g^{-1}}.$$

Since G is connected the image of Ad belongs to the connected component of $\mathrm{GL}(G_e)$ containing the identity, i.e. for each $g \in G$, $\det \mathrm{Ad}(g) > 0$. By Exercise A1 in

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[1, p. 147] we have

LEMMA 1. $d_gT = -d_eL_{g^{-1}} \circ d_gR_{g^{-1}} = -d_eR_{g^{-1}} \circ d_gL_{g^{-1}}$;

$$d_{(g_1,g_2)}\mu(X,Y) = d_{g_2}L_{g_1}(Y) + d_{g_1}R_{g_2}(X), \quad (X,Y) \in G_{g_1} \times G_{g_2}.$$

PROOF OF THEOREM 1. For each $g \in G$, we identify G_g with G_e by the differential of the left translation L_g at e. Let $I_g\colon G_g \to G_g$ be the identity map.

For any element in $[G, G]$, we can choose a representative $f\colon G \to G$ such that

(1) f is differentiable;

(2) f has only finitely many fixed points $g_1, \dots, g_k$;

(3) $\det(I_{g_i} - d_{g_i}f) \neq 0$.

Then a discussion in [2, p. 139] shows that

$$\operatorname{Lef}(f) = \sum_1^k \operatorname{sign}\det(I_{g_i} - d_{g_i}f).$$

$(f \cdot T)^{-1}(e) = \{g_1, \dots, g_k\}$, and for each i the differential of $(f \cdot T) \circ L_{g_i}$ at e is the composite

$$G_e \xrightarrow{dL_g} G_{g_i} \xrightarrow{d\Delta} G_{g_i} \times G_{g_i} \xrightarrow{df \times dT} G_{g_i} \times G_{g_i^{-1}} \xrightarrow{d\mu} G_e,$$

where $\Delta\colon G \to G \times G$ is the diagonal map. It follows from Lemma 1 that the above homomorphism is just the same as

$$\operatorname{Ad}(g^{-1})(d_e(L_{g_i^{-1}} \circ fL_{g_i}) - I_e)\colon G_e \to G_e.$$

Denote the map by A_i. Then by the convention (3) $\det A_i \neq 0$ and

$$(-1)^n \operatorname{sign}\det A_i = \operatorname{sign}\det(I_{g_i} - d_{g_i}f).$$

So we see that e is a regular value of $f \cdot T$ and

$$B(f) = \sum_1^k \operatorname{sign}\det A_i = (-1)^n \operatorname{Lef}(f).$$

This completes the proof.

To show the implications of the above theorem, recall that $H^*(G; F)$ is an exterior algebra $\Lambda(x_1, \dots, x_\lambda)$ generated by primitive elements x_i of odd degree [3] with $\lambda = \operatorname{rank} G$. Also from [3] we have

LEMMA 2. *If $f, f'\colon G \to G$ are two maps, and if $x \in H^*(G; F)$ is primitive, then $(f \cdot f')^*(x) = f^*(x) + f'^*(x)$.*

REMARK. It immediately follows from Lemma 2 that $\operatorname{Lef}(f \cdot f') = \operatorname{Lef}(f' \cdot f)$.

Since $\operatorname{Id} \cdot T\colon G \to G$ is the map collapsing G to unit e, by the lemma above $T^*(x_i) = -x_i$. Let $\cup$ be the cup product in $H^*(G; F)$. Then Lemma 2 also implies

$$\begin{aligned} B(f)^*(x_1 \cup \cdots \cup x_\lambda) &= (f \cdot T)^*(x_1 \cup \cdots \cup x_\lambda) \\ &= (f^*(x_1) - x_1) \cup \cdots \cup (f^*(x_\lambda) - x_\lambda). \end{aligned}$$

Since $n = \lambda \bmod 2$, we can rewrite Theorem 1 as follows:

THEOREM 2. $\operatorname{Lef}(f)x_1 \cup \cdots \cup x_\lambda = (x_1 - f^*(x_1)) \cup \cdots \cup (x_\lambda - f^*(x_\lambda))$.

Given a map $f\colon G \to G$ and an integer $k > 0$, let ${}^kf$ be the k-fold product of f defined inductively by ${}^1f = f$, ${}^kf = f \cdot {}^{k-1}f$.

COROLLARY 1. *For any map* $f\colon G \to G$, *there is an integer* k *with* $0 < k \le \lambda+1$ *such that* $\operatorname{Lef}({}^k f) \ne 0$.

PROOF. Given a map $h\colon G \to G$, regard the expression

$$(x_1 - h^*(x_1)) \cup \cdots \cup (x_\lambda - h^*(x_\lambda))$$

as a formal polynomial in the elements x_i and $h^*(x_j)$. For each integer t with $0 \le t \le \lambda$, there exists an element $A_t(h)$ in F such that $A_t(h)x_1 \cup \cdots \cup x_\lambda =$ sum of the monomials appearing in above polynomial, and containing just t elements in $h^*(x_j)$. Then Lemma 2 and Theorem 2 imply that

$$\operatorname{Lef}({}^k f) = \sum_0^\lambda k^t A_t(f) \quad \text{for any } k > 0.$$

So if $H = (a_{st})$ is the $(\lambda+1) \times (\lambda+1)$ Vandermonde matrix defined by $a_{st} = t^{s-1}$, $1 \le s, t \le \lambda+1$, then

$$(\operatorname{Lef}(f), \operatorname{Lef}({}^2 f), \ldots, \operatorname{Lef}({}^{\lambda+1} f)) = (A_0(f), A_1(f), \ldots, A_\lambda(f))H.$$

Now Corollary 1 follows from $\det H \ne 0$ and $A_0(f) = 1 \ne 0$.

Suppose $h\colon G \to G$ is a homomorphism. Then for any primitive element $x \in H^*(G; F)$, $h^*(x)$ is also primitive. Since the primitive elements form a submodule of $H^*(G; F)$ with basis $\{x_1, \ldots, x_\lambda\}$, there exists a $\lambda \times \lambda$ matrix M_h over F such that

$$(h^*(x_1), \ldots, h^*(x_\lambda)) = (x_1, \ldots, x_\lambda)M_h.$$

By Theorem 2 a discussion on linear algebra can show

COROLLARY 2. $\operatorname{Lef}(h) = \det(I - M_h)$.

Theorem 2 also leads to an easier way to make the following well-known computation:

COROLLARY 3. $\operatorname{Lef}({}^k \mathrm{Id}) = (1-k)^\lambda$; $\operatorname{Lef}({}^k T) = (1+k)^\lambda$.

The significance of it has been discussed in [**4**].

COMMENT. In fact Theorem 2 also has a purely cohomological proof without using Theorem 1. So all results, except Theorem 1, are valid for self-maps of a compact connected triangulable H-group.

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