PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 4, December 1988

ON STABILITY OF ENDOMORPHISMS

HIROSHI IKEDA

(Communicated by George R. Sell)

ABSTRACT. In this note we prove a generalization of R. Mañé's theorem. R. Mañé proved that C^r absolutely stable endomorphisms satisfy Axiom A. We prove that if an endomorphism f is both C^r structurally and infinitesimally stable, then f satisfies Axiom A.

1. Introduction. This note is a generalization of R. Mañé's theorem on absolute stability [**4**]. We prove this theorem under weaker hypotheses. Our Theorem is the following

THEOREM. *If $f \in \mathrm{End}^r(M)$ is both C^r structurally and infinitesimally stable, then f satisfies Axiom* A.

First we establish some background. Let M be a compact connected smooth manifold without boundary and let $\mathrm{End}^r(M)$, $r \geq 1$, be the space of C^r endomorphisms of M endowed with the C^r topology. We say that f is C^r structurally stable if there exists a neighborhood $\mathscr{U}$ of f in $\mathrm{End}^r(M)$ such that for every $g \in \mathscr{U}$ there exists a homeomorphism h of M satisfying $fh = hg$. If $f \in \mathrm{End}^r(M)$, let $\mathrm{Per}(f)$ be the set of all periodic points of f and let $\Omega(f) = \{x \in M |$ for every neighborhood U of x, there exists $n > 0$ with $f^n(U) \cap U \neq \varnothing\}$. We denote by $S(f)$ the set of singularities of f, i.e. those points x of M where $Tf|T_xM$ is not injective.

We give two definitions following R. Mañé [**4**].

DEFINITION. We say that $f \in \mathrm{End}^r(M)$ is C^r *absolutely stable* if there exist a neighborhood $\mathscr{U}$ of f in $\mathrm{End}^r(M)$ and a constant $K > 0$ such that for all $g \in \mathscr{U}$ there exists a homeomorphism h of M satisfying $gh = hf$ and $d(h, I) \leq Kd(f, g)$ where $d(\cdot,\cdot)$ is defined by $d(f_1, f_2) = \sup\{\rho(f_1(x), f_2(x)) | x \in M\}$, $\rho(\cdot,\cdot)$ being a metric on M and I is the identity map of M.

DEFINITION. We say that $f \in \mathrm{End}^r(M)$ satisfies *Axiom* A if there exist a continuous splitting $TM|\Omega(f) = E^s \oplus E^u$, and a Riemannian metric $|\ |$ on M, and constants $K > 0$, $0 < \lambda < 1$ satisfying:

(a) $(Tf)E^s \subset E^s$, $(Tf)E^u = E^u$;

(b)
$$|(Tf)^n|E^s_x| \leq K\lambda^n \quad \text{for } x \in \Omega(f),\ n > 0,$$
$$|(Tf)^n v| \geq K\lambda^{-n}|v| \quad \text{for } x \in \Omega(f),\ v \in E^u_x,\ n > 0;$$

(c) if $x_1 \neq x_2 \in \Omega(f)$ and $f(x_1) = f(x_2) = x$, then $E^s_x = \{0\}$;

(d) $\mathrm{Per}(f)$ is dense in $\Omega(f)$;

(e) $S(f) \cap \Omega(f) = \varnothing$.

In [**8**] the following conjecture was stated.

Received by the editors September 29, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58F10; Secondary 58F15.
Key words and phrases. Infinitesimal stability, Axiom A, prehyperbolic set, $\overline{P}$-stability.

CONJECTURE. If $f \in \mathrm{End}^r(M)$ is C^r structurally stable, then $\Omega(f) \cap S(f) = \emptyset$.

In fact, $\mathrm{Per}(f) \cap S(f) = \emptyset$ holds if f is C^r structurally stable, and $\Omega(f) \cap S(f) = \emptyset$ holds if f is C^r absolutely stable. The latter property is a part of the theorem by R. Mañé [**4**].

THEOREM M. *C^r absolutely stable endomorphisms satisfy Axiom* A.

On the other hand, F. Przytycki defined Axiom A somewhat different from Mañé's Axiom A, and found sufficient conditions for Ω-stability of an endomorphism [**6**, **7**].

Our Theorem is a generalization of Theorem M.

2. Proof of the Theorem. In this section we will prove our Theorem. To prove the Theorem we need some definitions and lemmas. If $\Lambda \subset M$ is a compact subset of M let $\Gamma^b(\Lambda)$ be the space of bounded sections of $TM|\Lambda$ with the norm $||\eta|| = \sup\{||\eta(x)|| \,|x \in \Lambda\}$ and let $\Gamma^0(\Lambda)$ be the closed subspace of continuous sections. If $f \in \mathrm{End}^r(M)$ and $f(\Lambda) \subset \Lambda$ let $T_fM|\Lambda$ be the vector bundle on Λ consisting of couples (p, v) with $p \in \Lambda$, $v \in T_{f(p)}M$. Let $\Gamma^b_f(\Lambda)$, $\Gamma^0_f(\Lambda)$ be the corresponding spaces of bounded and continuous sections of $T_fM|\Lambda$.

We define the linear operator L_f: $\Gamma^b(\Lambda) \to \Gamma^b_f(\Lambda)$ by

$$L_f(\eta) = (Tf) \circ \eta - \eta \circ f \quad \text{for } \eta \in \Gamma^b(\Lambda).$$

DEFINITION. Let $f \in \mathrm{End}^r(M)$ and let $\Lambda \subset M$ be a compact subset with $f(\Lambda) = \Lambda$. We say that Λ is a *prehyperbolic set* for f if there exist a continuous splitting $TM|\Lambda = E^s \oplus E^u$, and a Riemannian metric $|\ |$ on M and constants $K > 0$, $0 < \lambda < 1$ satisfying:

(a) $(Tf)E^s \subset E^s$, $(Tf)E^u = E^u$;

(b) $$|(Tf)^n|E^s_x| \le K\lambda^n \text{ for } x \in \Lambda, n > 0,$$
$$|(Tf)^n v| \ge K\lambda^{-n}|v| \text{ for } x \in \Lambda,\ v \in E^u_x, n > 0;$$

(c) if $x_1 \ne x_2 \in \Lambda$ and $f(x_1) = f(x_2) = x$, then $E^s_x = \{0\}$.

We say that $f \in \mathrm{End}^r(M)$ is *infinitesimally stable* if the linear operator L_f: $\Gamma^0(M) \to \Gamma^0_f(M)$ is surjective. It is evident that the absolute stability of f implies the structural stability of f. Also it is easy to prove that if $f \in \mathrm{End}^r(M)$ is absolutely stable, then f is infinitesimally stable by the similar argument for the case of diffeomorphisms in [**1**]. If $x \in M$ let $\omega(x) = \omega(x, f)$ be the set of ω-limit points of x for f and let $\overline{L}^+(f)$ be the closure of $L^+(f) = \{\omega(x) | x \in M\}$.

In the proof of the theorem, we shall use the following two lemmas of [**4**].

LEMMA 1. *If $f \in \mathrm{End}^r(M)$ is infinitesimally stable, then $\overline{L}^+(f)$ is prehyperbolic.*

LEMMA 2. *Suppose that $f \in \mathrm{End}^r(M)$ and $\overline{L}^+(f)$ is prehyperbolic. Then given $x \in \overline{L}^+(f)$ and a neighborhood $\mathscr{U}$ of f there exist a neighborhood U of x and $g \in \mathscr{U}$ such that:*

(a) *$g(y) = f(y)$ for all $y \in U$;*

(b) *$x \in \mathrm{Per}(g)$.*

From the argument of R. Mañé [**4**], it clearly follows that we only need to prove the following proposition in order to prove the Theorem.

PROPOSITION. *If $f \in \mathrm{End}^r(M)$ is both C^r structurally and infinitesimally stable, then $\overline{L}^+(f) = \overline{\mathrm{Per}(f)}$.*

To prove the Proposition, we introduce the concept of $\overline{P}$-stability. $\overline{P} = \overline{\mathrm{Per}(f)}$ will denote the closure of the set of periodic points of f. We say that $f \in \mathrm{End}^r(M)$ is $\overline{P}$-stable if there exists a neighborhood $\mathscr{U}$ of f in $\mathrm{End}^r(M)$ such that for each $g \in \mathscr{U}$ there exists a homeomorphism $h: \overline{\mathrm{Per}(f)} \to \overline{\mathrm{Per}(g)}$ satisfying $hf = gh$ [**5**]. It is obvious that the structural stability implies the $\overline{P}$-stability.

PROOF OF PROPOSITION. We first suppose that $\overline{\mathrm{Per}(f)} \subsetneq \overline{L}^+(f)$. We show that $\overline{L}^+(f) \cap S(f) = \varnothing$. Suppose that $x \in \overline{L}^+(f) \cap S(f)$. By Lemmas 1 and 2, we can take $g \in \mathrm{End}^r(M)$ nearby f coinciding with f in a neighborhood of x (therefore $x \in S(g)$) and such that $x \in \mathrm{Per}(g)$. Hence $x \in \mathrm{Per}(g) \cap S(g)$. This contradicts the stability of g. Hence $\overline{L}^+(f) \cap S(f) = \varnothing$. Also it is obvious that $\overline{\mathrm{Per}(f)}$ is prehyperbolic. Therefore $\overline{\mathrm{Per}(f)}$ has a decomposition, $\Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_n$ into disjoint prehyperbolic sets, following the method of Przytycki [**7**] and Newhouse [**5**]. Here each Λ_i is a prehyperbolic set such that the periodic points are dense in Λ_i. Let U_i be a compact neighborhood of Λ_i for $1 \le i \le n$. If we choose U_i small enough, we have that $U_i \cap U_j = \varnothing$ if $i \ne j$ and $\overline{L}^+(f) \setminus \bigcup_{1 \le i \le n} U_i \ne \varnothing$.

Now suppose that $f|\Lambda_i$ is injective. Then shrinking U_i if necessary, $f|U_i$ is a diffeomorphism onto its image. By Theorem 7.3 of [**2**], there is a neighborhood $\mathscr{N}_i$ of f in $\mathrm{End}^r(M)$ such that if $g \in \mathscr{N}_i$ then there exists a homeomorphism h close to identity which conjugates Λ_i with a g-invariant subset of U_i.

Next suppose that $f|\Lambda_i$ is not injective. Then it is easy to show that $f|\Lambda_i$ is a (quasi-) expanding map, i.e. $\dim E^u_x = \dim M$ for any $x \in \Lambda_i$. By the similar argument to Przytycki [**7**], there exists a neighborhood $\mathscr{N}_i$ of f in $\mathrm{End}^r(M)$ such that if $g \in \mathscr{N}_i$ then there exists a homeomorphism h from Λ_i onto its image satisfying $gh = hf$ and $h(\Lambda_i) \subset U_i$.

From above arguments and $\overline{P}$-stability of f, it follows that there exists a neighborhood $\mathscr{N}$ of f in $\mathrm{End}^r(M)$ such that $\overline{\mathrm{Per}(g)} \subset \bigcup_{1 \le i \le n} U_i$ for any $g \in \mathscr{N}$. Let $x \in \overline{L}^+(f) \setminus \bigcup_{1 \le i \le n} U_i$. By Lemma 2, there exists $g \in \mathscr{N}$ such that x is a periodic point of g. This is a contradiction.

Subsequently we obtain the Corollary of the Theorem:

COROLLARY. *If $f \in \mathrm{End}^r(M)$ is both C^r structurally and infinitesimally stable, then f satisfies Axiom* A *and no cycle condition.*

REFERENCES

1. J. Guckenheimer, *Absolutely Ω-stable diffeomorphisms*, Topology **11** (1972), 195–197.
2. M. Hirsch and C. Pugh, *Stable manifolds and hyperbolic sets*, Proc. Sympos. Pure Math., vol. 14, Global Analysis, Amer. Math. Soc., Providence, R.I., 1970, pp. 133–163.
3. R. Mañé, *On infinitesimal and absolute stability of diffeomorphisms*, Dynamical Systems (Warwick, 1974), Lecture Notes in Math., vol. 468, Springer-Verlag, Berlin, Heidelberg and New York, pp. 151–161.
4. ——, *Axiom* A *for endomorphisms*, Lecture Notes in Math., vol. 597, Springer-Verlag, Berlin, Heidelberg and New York, pp. 379–388.
5. S. Newhouse, *Hyperbolic limit sets*, Trans. Amer. Math. Soc. **67** (1972), 125–150.
6. F. Przytycki, *Anosov endomorphisms*, Studia Math. **58** (1976), 249–285.

7. ——, *On Ω-stability and structural stability of endomorphisms satisfying Axiom* A, Studia Math. **60** (1977), 61–77.
8. C. Pugh and R. Mañé, *Stability of endomorphisms*, Lecture Notes in Math., vol. 468, Springer-Verlag, Berlin, Heidelberg and New York, 1975, pp. 175–184.

DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE AND ENGINEERING, WASEDA UNIVERSITY, SHINJUKU, TOKYO, JAPAN