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ALTERNATING SEQUENCES WITH NONPOSITIVE OPERATORS

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ABSTRACT. Let (T_n) be a sequence of linear operators acting on complex valued functions on a σ-finite measure space. Assume that each T_n contracts all the p-norms, $1 \leq p \leq \infty$ (i.e. $\|T_n\|_p \leq 1$). It is shown that a.e. $\lim_n T_1^* \cdots T_n^* T_n \cdots T_1 f$ exists for each L_p function f, $1 < p < \infty$.

1. Introduction. Let L_p, $1 \leq p \leq \infty$, be the usual Banach Spaces of complex valued functions on a σ-finite measure space. Let (T_n) be a sequence of linear operators that are contractions of all L_p spaces, $1 \leq p \leq \infty$, simultaneously. We are going to show that $T_1^* \cdots T_n^* T_n \cdots T_1 f$ converges a.e. for each f in L_p, $1 < p < \infty$.

The following result is a part of a theorem due to Starr [**6**]; it generalizes some parts of the earlier related results of Rota [**5**] and Stein [**7**]. (See [**4**, **8**, and **3**] for further references.)

(1.1) THEOREM. *If $1 < p < \infty$ and if $f \in L_p$ then*

$$\left\| \sup_n |T_1^* \cdots T_n^* T_n \cdots T_1 f| \right\|_p \leq \frac{p}{p-1} \|f\|_p. \tag{1.2}$$

If, furthermore, each T_n is also a positive operator then

$$\lim_n T_1^* \cdots T_n^* T_n \cdots T_1 f \tag{1.3}$$

exists a.e.

The existence of (1.3) in the general nonpositive case has apparently remained open, as the dominated estimate in (1.2) does not seem to be sufficient to establish this convergence. In this note we would like to show that an extension of the techniques used in [**3**] gives another dominated estimate (Theorem (2.1) below), which implies the existence of (1.3) in the general nonpositive case.

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2. The main result. We are going to obtain the following result.

(2.1) THEOREM. *Given p, $1 < p < \infty$, and $\varepsilon > 0$, there is a $\delta = \delta(p, \varepsilon) > 0$, depending only on p and ε, with the following property. Let (T_n) be a sequence of linear operators that are contractions of all L_p spaces, $1 \leq p \leq \infty$, simultaneously. Let $f \in L_p$ and*

$$f_n = T_1^* \cdots T_n^* T_n \cdots T_1 f. \tag{2.2}$$

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Then

$$\left\| \sup_n |f_n - f| \right\|_p < \varepsilon \|f\|_p \tag{2.3}$$

whenever

$$\|f\|_p - \|T_n \cdots T_1 f\|_p < \delta \|f\|_p \tag{2.4}$$

for all n.

This result is quite similar to an estimate in [**3**] (called Estimate B in that paper). First, it is easy to see that this theroem implies the a.e. convergence of f_n; complete details are given in [**3**]. Also note that, since this is a uniform estimate, it is enough to prove it in the finite dimensional case, i.e. under the assumption that the underlying measure space contains only finitely many points. Complete details of the passage from the finite dimensional case are again given in [**3**].

The proof of Theorem (2.1) in the finite dimensional case proceeds along the same lines as the corresponding proof in [**3**]. Given a sequence of operators T_n, one constructs a general (nonatomic) measure space (Z, μ) and represents the sequence (T_n) by a sequence of isometries, induced by a sequence of point transformations (τ_n) of Z. In the present case there is even an important simplification. Since T_n is a contraction of all L_p spaces, the associated point transformation τ_n is measure preserving. There are, however, also some complications. First, since T_n is not positive, it will be represented by a complex isometry that will involve both the point transformation τ_n and the multiplication by a function ϕ_n of unit modulus. Also, in [**3**] it was possible to assume that the norm of T_n is exactly 1, by multiplying T_n, if necessary, by a constant greater than 1. In the present case this is impossible, as we want to have T_n as a contraction of all L_p spaces. This seems to necessitate the construction of Z as an infinite measure space, rather than a space of finite measure as in [**3**].

Finally we note there are also some similarities between the present arguments and the ideas involved in [**1**] and in [**2**].

3. Preliminaries. Let d be a fixed integer, $d \geq 1$. The indices i and j will range through the integers $1, 2, \ldots, d$. Consider a measure space consisting of d points with masses m_i, $m_i > 0$. Let l_p, $1 \leq p \leq \infty$, denote the usual complex L_p-spaces associated with this measure space. Hence each l_p is the d-dimensional complex space $\mathbf{C}^d$ with the corresponding p-norm.

Let $T\colon \mathbf{C}^d \to \mathbf{C}^d$ be a linear operator which is a contraction of all l_p spaces, similtaneously. Represent T by a matrix (T_{ij}) such that $(Tx)_i = \sum_j T_{ij} x_j$, $x = (x_i) \in \mathbf{C}^d$. Let $H_{ij} = |T_{ij}|$ and let $H\colon \mathbf{C}^d \to \mathbf{C}^d$ denote the corresponding operator, $(Hx)_i = \sum_j H_{ij} x_j$. Then it is easy to see that H is also a contraction of all l_p spaces and also that

$$r_j = \frac{1}{m_j} \sum_i H_{ij} m_i \leq 1, \tag{3.1}$$

$$s_i = \sum_j H_{ij} \leq 1. \tag{3.2}$$

These inequalities express the facts that H is an l_1- l_∞-contraction, respectively. Finally observe that

$$\sigma = \sum_j (1 - r_j) m_j = \sum_i (1 - s_i) m_i. \tag{3.3}$$

The linear operators T and H will now be represented by an invertible measure preserving transformation τ of a σ-finite nonatomic measure space (Z, μ), as follows.

(3.4) DEFINITION OF Z. The subscripts i and j range through the integers $1, 2, \ldots, d$, as before. The superscript k will range through the set of all integers $0, \pm 1, \pm 2, \ldots$. The measure space (Z, μ) will consist of a part of the xy-plane, together with the two dimensional Lebesgue measure. The length of an interval I is denoted by $l(I)$.

Let I_i^k be the disjoint intervals on the x-axis and J_j^k disjoint intervals on the y-axis such that $l(I_i^k) = m_i$ and $l(J_j^k) = 1$. We then let

$$Z = \bigcup_k \bigcup_i I_i^k \times J_i^k. \tag{3.5}$$

The points of Z are denoted by z or by (x, y), as usual. A function f defined on Z is said to be constant on vertical lines if f depends only on the x-coordinate of $(x, y) \in Z$. A subset G of Z is called a vertical set if its characteristic function χ_G is constant on vertical lines. Functions on Z that are constant on horizontal lines and horizontal subsets of Z are defined similarly. Note that each rectangle $I_i^k \times J_i^k$ is both a vertical and a horizontal subset of Z.

We will deal with partitions of Z into finitely many atoms, which will contain atoms of infinite measure. The conditional expectation with respect to such a partition $\mathscr{G}$ will be denoted by $E(\cdot|\mathscr{G})$ and defined as zero on atoms of infinite measure. In particular $\mathscr{P}$ will denote the partition of Z into $d+2$ atoms, consisting of d atoms $P_i = I_i^0 \times J_i^0$ of finite measure and 2 atoms

$$Z^- = \bigcup_{k<0} \bigcup_i I_i^k \times J_i^k, \qquad Z^+ = \bigcup_{k>0} \bigcup_i I_i^k \times J_i^k$$

of infinite measure. The condition expectation with respect to $\mathscr{P}$ will also be denoted by E. Note that there is a natural positive isometric isomorphism between l_p and $EL_p(Z, \mu)$. We will not write this isomorphism explicitly and treat these two spaces as identical.

(3.6) AUTOMORPHISMS OF Z. All automorphisms of Z considered in this paper are invertible and measure preserving transformation $\tau\colon Z \to Z$ of the following special type. Each rectangle $I_j^k \times J_j^k$ is partitioned into $2d$ horizontal rectangles R_{ij}^k and $\tilde{R}_{ij}^k$, some of them possibly of zero measure. Also, each rectangle $I_i^k \times J_i^k$ is partitioned into $2d$ vertical rectangles S_{ij}^k and $\tilde{S}_{ij}^k$ such that $\mu(R_{ij}^k) = \mu(S_{ij}^k)$ and $\mu(\tilde{R}_{ij}^k) = \mu(\tilde{S}_{ij}^k)$ and such that these measures depend only on i and j, but not on k. Then τ maps R_{ij}^k and $\tilde{R}_{ij}^k$ onto S_{ij}^k and $\tilde{S}_{ij}^{k+1}$, respectively, and the restriction of τ to each R_{ij}^k or $\tilde{R}_{ij}^k$ is of the form

$$\tau(x, y) = (ax + b, cy + d),$$

where a, b, c, d are four constants depending on R_{ij}^k or $\tilde{R}_{ij}^k$.

Let Q be the induced operator on functions defined by

$$(Qf)(z) = f(\tau^{-1}z), \qquad z \in Z,\ f\colon Z \to \mathbf{C}.$$

It is easy to see that one can always choose the measures of R_{ij}^k and $\tilde{R}_{ij}^k$ in such a way that the operator H is represented as

$$H = EQE. \tag{3.7}$$

In fact it is enough to take $\mu(R_{ij}^k) = \mu(S_{ij}^k) = H_{ij}m_i$, and define $\mu(\tilde{R}_{ij}^k) = \mu(\tilde{S}_{ij}^k)$ rather arbitrarily, subject only to the requirements that

$$\sum_i (\mu(R_{ij}^k) + \mu(\tilde{R}_{ij}^k)) = m_j \quad \text{and} \quad \sum_j (\mu(S_{ij}^k) + \mu(\tilde{S}_{ij}^k)) = m_i.$$

Note that if $\sigma = 0$, as defined in (3.3), then we must take $\mu(\tilde{R}_{ij}^k) = \mu(\tilde{S}_{ij}^k) = 0$. Otherwise we may let, for example,

$$\mu(\tilde{R}_{ij}^k) = \mu(\tilde{S}_{ij}^k) = (1/\sigma)(1 - s_i)(1 - r_j)m_i m_j.$$

We now observe that if Q represents H, as in (3.7), then we can also represent the original operator T as

$$T = E\phi QE = EKE \tag{3.8}$$

where $\phi\colon Z \to \mathbf{C}$ is a function with constant unit modulus everywhere and $K = \phi Q$ denotes the operator on functions defined by $Kf = \phi(Qf)$. Furthermore one can choose ϕ as a function measurable with respect to the partition $\mathscr{P} \vee \tau\mathscr{P}$. In fact, we can define ϕ as follows.

$$\phi(z) = \begin{cases} (T_{ij}/H_{ij}) & \text{if } z \in S_{ij}^0 \text{ and } \mu(S_{ij}^0) > 0, \\ 1 & \text{otherwise.} \end{cases}$$

Note that both Q and K are invertible isometries of all $L_p(Z, \mu)$ spaces. Hence we see that $K^* = K^{-1}$.

(3.9) LEMMA. *Let a function f be constant on vertical lines and zero on Z^-. Let $\mathscr{G}$ be a partition consisting of vertical atoms and refining $\mathscr{P}$. Then $E(f|\mathscr{G}) = E(f|\mathscr{G} \vee \tau^{-1}\mathscr{P})$.*

PROOF. Both of these expectations are zero on Z^-, since Z^- is a union of some atoms of $\mathscr{G}$ and $f = 0$ on Z^-. Also, since $\tau^{-1}Z^+ \supset Z^+$ the restrictions of $\mathscr{G}$ and $\mathscr{G} \vee \tau^{-1}\mathscr{P}$ to Z^+ are the same. On the remaining part of Z each atoms G of $\mathscr{G}$ has finite measure and is a vertical subset of one of the rectangles P_j. In $\mathscr{G} \vee \tau^{-1}\mathscr{P}$, this atom G is further divided by the horizontal sets R_{ij}^0 and $\bigcup_i \tilde{R}_{ij}^0$. Since f is constant on vertical lines, the average value of f on each of these pieces is the same and is equal to the average value of f on G. This shows that the two expectations in the lemma are the same on G.

(3.10) LEMMA. *Let a function f be constant on horizontal lines and zero on Z^+. Let $\mathscr{G}$ be a partition consisting of horizontal atoms and refining $\mathscr{P}$. Then $E(f|\mathscr{G}) = E(f|\mathscr{G} \vee \tau\mathscr{P})$.*

The proof is similar to the one above and is omitted.

(3.11) LEMMA. *Let f be constant on vertical lines and zero on Z^-. Let $\mathscr{G}$ be a vertical refinement of $\mathscr{P}$. Then*

$$KE(f|\mathscr{G}) = E(Kf|\mathscr{P} \vee \tau\mathscr{G}).$$

PROOF. First note that, since τ is an invertible and measure preserving transformation,

$$\begin{aligned} QE(f|\mathscr{G}) &= E(f|\mathscr{G}) \circ \tau^{-1} \\ &= E(f \circ \tau^{-1}|\tau\mathscr{G}) = E(Qf|\tau\mathscr{G}), \end{aligned}$$

for any f and $\mathscr{G}$. Under the hypotheses of the lemma we also have, by (3.9),

$$E(f|\mathscr{G}) = E(f|\mathscr{G} \vee \tau^{-1}\mathscr{P}).$$

Hence

$$\begin{aligned} KE(f|\mathscr{G}) &= \phi QE(f|\mathscr{G} \vee \tau^{-1}\mathscr{P}) = \phi E(Qf|\mathscr{P} \vee \tau\mathscr{G}) \\ &= E(\phi Qf|\mathscr{P} \vee \tau\mathscr{G}) = E(Kf|\mathscr{P} \vee \tau\mathscr{G}), \end{aligned}$$

where the third equality follows from the facts that ϕ is $\mathscr{P} \vee \tau\mathscr{P}$-measurable and that $\mathscr{P} \vee \tau\mathscr{G}$ is finer than $\mathscr{P} \vee \tau\mathscr{P}$.

(3.12) LEMMA. *Let f be constant on horizontal lines and zero on Z^+. Then*

$$K^{-1}Ef = E(K^{-1}f|\mathscr{P} \vee \tau^{-1}\mathscr{P}).$$

PROOF. Since $Ef = E(f|\mathscr{P}) = E(f|\mathscr{P} \vee \tau\mathscr{P})$ by Lemma (3.10),

$$\begin{aligned} K^{-1}Ef &= Q^{-1}\phi^{-1}E(f|\mathscr{P} \vee \tau\mathscr{P}) \\ &= Q^{-1}E(\phi^{-1}f|\mathscr{P} \vee \tau\mathscr{P}) \\ &= E(K^{-1}f|\mathscr{P} \vee \tau^{-1}\mathscr{P}). \end{aligned}$$

4. The main argument. Let T_m, $m = 1, \ldots, n$, be n operators on a d-dimensional space, that contract all l_p-norms, $1 \leq p \leq \infty$, as in the previous section. Let (Z, μ) be the measure space, as constructed above, and let τ_m be the invertible measure preserving automorphisms of Z, associated with T_m, so that $T_m = EK_mE$, $K_m = \phi_m Q_m$, with the corresponding $L_p(Z, \mu)$-isometries K_m and Q_m, as before.

(4.1) LEMMA. *$T_m \cdots T_1 Ef = EK_m \cdots K_1 Ef$, for any $m = 1, \ldots, n$.*

PROOF. The assertion is correct for $m = 1$. To prove the general case, first note that if a function g satisfies the hypotheses of Lemma (3.11), then $K_m g$ also satisfies these hypotheses, for any m. Hence we see that $K_m \cdots K_1 Ef$ satisfies these hypotheses. Then

$$K_{m+1}E(K_m \cdots K_1 Ef) = E(K_{m+1}K_m \cdots K_1 Ef|\mathscr{P} \vee \tau_{m+1}\mathscr{P}),$$

by Lemma (2.11). Since

$$EE(\cdot|\mathscr{P} \vee \tau_{m+1}\mathscr{P}) = E,$$

we have that

$$EK_{m+1}EK_m \cdots K_1 Ef = EK_{m+1}K_m \cdots K_1 Ef.$$

This enables us to complete the induction step.

(4.2) LEMMA. *$T_1^* \cdots T_m^* Ef = EK_1^{-1} \cdots K_m^{-1} Ef$ for any $m = 1, \ldots, n$.*

We omit the proof as it is similar to the proof above.

(4.3) LEMMA. *Starting with a function f define a sequence $g_0, g_1, \ldots, g_n$ as*

$$\begin{aligned} g_0 &= K_n \cdots K_1 Ef, \\ g_m &= K_n \cdots K_{m+1} E K_m \cdots K_1 Ef, \qquad 1 \le m < n, \\ g_n &= E K_n \cdots K_1 Ef. \end{aligned}$$

Then there is a monotone sequence of partitions $\mathscr{G}_0 > \cdots > \mathscr{G}_n$ such that $g_m = E(g_0 | \mathscr{G}_m)$, $0 \le m \le n$.

PROOF. It is clear that $\mathscr{G}_n = \mathscr{P}$. If $1 \le m < n$, let $h = K_m \cdots K_1 Ef$. Then h is a function that satisfies the hypotheses of Lemma (3.11). Hence we see that

$$\begin{aligned} g_m &= K_n \cdots K_{m+1} Eh \\ &= K_n \cdots K_{m+2} E(K_{m+1} h | \mathscr{P} \vee \tau_{m+1} \mathscr{P}) \\ &\quad E(K_n \cdots K_{m+1} h | \mathscr{P} \vee \tau_n \mathscr{P} \vee \tau_n \tau_{n-1} \mathscr{P} \vee \cdots \vee \tau_n \cdots \tau_{m+1} \mathscr{P}) \\ &= E(g_0 | \mathscr{G}_m), \end{aligned}$$

with

$$\mathscr{G}_m = \mathscr{P} \vee \tau_n \mathscr{P} \vee \cdots \vee \tau_n \cdots \tau_{m+1} \mathscr{P},$$

by $(n - m)$ applications of Lemma (3.11).

(4.4) LEMMA. *With the notations of the previous lemma,*

$$\left\| \max_{0 \le m \le n} |g_m - g_n| \right\|_p \le \frac{p}{p-1} \|g_0 - g_n\|_p,$$

and

$$\left\| \max_{0 \le m \le n} |g_m - g_0| \right\|_p \le \frac{2p}{p-1} \|g_0 - g_n\|_p,$$

for any $1 < p < \infty$.

PROOF. The sequence $(g_m - g_n)$, $m = 0, \ldots, n$, is a martingale, since

$$g_m - g_n = E(g_0 - g_n | \mathscr{G}_m)$$

and $\mathscr{G}_0 > \mathscr{G}_1 > \cdots > \mathscr{G}_n$. Hence the first inequality follows from the martingale maximal theorem. The second inequality is an obvious consequence of the first inequality.

(4.5) THEOREM. *Given p, $1 < p < \infty$, and $\varepsilon > 0$, there is a $\delta = \delta(p, \varepsilon) > 0$, depending only on p and ε, such that if*

$$\|Ef\|_p - \|T_n \cdots T_1 Ef\|_p < \delta \|Ef\|_p$$

then

$$\left\| \max_{1 \le m \le n} |f_m - Ef| \right\|_p < \varepsilon \|Ef\|_p,$$

with

$$f_m = T_1^* \cdots T_m^* T_m \cdots T_1 Ef, \qquad 1 \le m \le n.$$

PROOF. Let $V = K_1^{-1} \cdots K_n^{-1}$. Then we see that

$$\begin{aligned} f_m &= EK_1^{-1} \cdots K_m^{-1} EK_m \cdots K_1 Ef \\ &= EVK_n \cdots K_{m+1} EK_m \cdots K_1 Ef \\ &= EVg_m. \end{aligned}$$

Hence

$$f_m - Ef = EV(g_m - g_0).$$

Since V is an invertible isometry of each $L_p(Z, \mu)$ space, $1 \le p \le \infty$, we see easily that

$$\begin{aligned} \left\| \max_{1 \le m \le n} |f_m - Ef| \right\|_p &\le \left\| \max_{1 \le m \le n} |g_m - g_0| \right\|_p \\ &\le \frac{2p}{p-1} \|g_0 - g_n\|_p, \end{aligned}$$

where the second inequality follows from Lemma (4.4).

Now g_n is the condition expectation of g_0 with respect to a partition $\mathscr{G}_n$. Hence, as observed in [**3**], for each $\xi > 0$, there is an $\eta > 0$, depending only on ξ and p, such that if

$$\|g_0\|_p - \|g_n\|_p < \eta \|g_0\|_p$$

then

$$\|g_0 - g_n\|_p < \xi \|g_0\|_p.$$

But

$$\|g_0\|_p = \|K_n \cdots K_1 Ef\|_p = \|Ef\|_p$$

and

$$\|g_n\|_p = \|EK_n \cdots K_1 Ef\|_p = \|T_n \cdots T_1 Ef\|_p.$$

Hence it is enough to choose $\xi > 0$ sufficiently small so that

$$\frac{2p}{p-1} \xi < \varepsilon$$

and then take δ to be equal to the corresponding $\eta > 0$.

An observed earlier, the main result, Theorem (2.1), follows easily from the result above.

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