ALTERNATING SEQUENCES WITH NONPOSITIVE OPERATORS

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(Communicated by W. J. Davis)

ABSTRACT. Let $(T_n)$ be a sequence of linear operators acting on complex valued functions on a $\sigma$-finite measure space. Assume that each $T_n$ contracts all the $p$-norms, $1 \leq p \leq \infty$ (i.e. $\|T_n\|_p \leq 1$). It is shown that a.e. $\lim_n T_1^* \cdot \cdots \cdot T_n^* T_n^* \cdots T_1 f$ exists for each $L_p$ function $f$, $1 < p < \infty$.

1. Introduction. Let $L_p$, $1 \leq p \leq \infty$, be the usual Banach Spaces of complex valued functions on a $\sigma$-finite measure space. Let $(T_n)$ be a sequence of linear operators that are contractions of all $L_p$ spaces, $1 \leq p \leq \infty$, simultaneously. We are going to show that $T_1^* \cdot \cdots \cdot T_n^* T_n^* \cdots T_1 f$ converges a.e. for each $f$ in $L_p$, $1 < p < \infty$.

The following result is a part of a theorem due to Starr [6]; it generalizes some parts of the earlier related results of Rota [5] and Stein [7]. (See [4, 8, and 3] for further references.)

\[
\text{(1.1) Theorem. If } 1 < p < \infty \text{ and if } f \in L_p \text{ then }
\sup_n |T_1^* \cdot \cdots \cdot T_n^* T_n^* \cdots T_1 f|_p \leq \frac{p}{p-1} \|f\|_p.
\]

If, furthermore, each $T_n$ is also a positive operator then

\[
\lim_n T_1^* \cdot \cdots \cdot T_n^* T_n^* \cdots T_1 f
\]

exists a.e.

The existence of (1.3) in the general nonpositive case has apparently remained open, as the dominated estimate in (1.2) does not seem to be sufficient to establish this convergence. In this note we would like to show that an extension of the techniques used in [3] gives another dominated estimate (Theorem (2.1) below), which implies the existence of (1.3) in the general nonpositive case.

The author would like to thank Dr. I. Assani for bringing this problem to his attention.

2. The main result. We are going to obtain the following result.

\[
\text{(2.1) Theorem. Given } p, 1 < p < \infty, \text{ and } \varepsilon > 0, \text{ there is a } \delta = \delta(p, \varepsilon) > 0, \text{ depending only on } p \text{ and } \varepsilon, \text{ with the following property. Let } (T_n) \text{ be a sequence of linear operators that are contractions of all } L_p \text{ spaces, } 1 \leq p \leq \infty, \text{ simultaneously. Let } f \in L_p \text{ and }
\]

\[
f_n = T_1^* \cdot \cdots \cdot T_n^* T_n^* \cdots T_1 f.
\]
Then
\[(2.3) \quad \left\| \sup_n |f_n - f| \right\|_p < \varepsilon \|f\|_p\]
whenever
\[(2.4) \quad \|f\|_p - \|T_n \cdots T_1 f\|_p < \delta \|f\|_p\]
for all \(n\).

This result is quite similar to an estimate in [3] (called Estimate B in that paper). First, it is easy to see that this theorem implies the a.e. convergence of \(f_n\); complete details are given in [3]. Also note that, since this is a uniform estimate, it is enough to prove it in the finite dimensional case, i.e. under the assumption that the underlying measure space contains only finitely many points. Complete details of the passage from the finite dimensional case are again given in [3].

The proof of Theorem (2.1) in the finite dimensional case proceeds along the same lines as the corresponding proof in [3]. Given a sequence of operators \(T_n\), one constructs a general (nonatomic) measure space \((Z, \mu)\) and represents the sequence \((T_n)\) by a sequence of isometries, induced by a sequence of point transformations \((r_n)\) of \(Z\). In the present case there is even an important simplification. Since \(T_n\) is a contraction of all \(L_p\) spaces, the associated point transformation \(r_n\) is measure preserving. There are, however, also some complications. First, since \(T_n\) is not positive, it will be represented by a complex isometry that will involve both the point transformation \(r_n\) and the multiplication by a function \(\phi_n\) of unit modulus. Also, in [3] it was possible to assume that the norm of \(T_n\) is exactly 1, by multiplying \(T_n\), if necessary, by a constant greater than 1. In the present case this is impossible, as we want to have \(T_n\) as a contraction of all \(L_p\) spaces. This seems to necessitate the construction of \(Z\) as an infinite measure space, rather than a space of finite measure as in [3].

Finally we note there are also some similarities between the present arguments and the ideas involved in [1] and in [2].

3. Preliminaries. Let \(d\) be a fixed integer, \(d \geq 1\). The indices \(i\) and \(j\) will range through the integers \(1, 2, \ldots, d\). Consider a measure space consisting of \(d\) points with masses \(m_i, m_i > 0\). Let \(l_p, 1 \leq p \leq \infty\), denote the usual complex \(L_p\)-spaces associated with this measure space. Hence each \(l_p\) is the \(d\)-dimensional complex space \(\mathbb{C}^d\) with the corresponding \(p\)-norm.

Let \(T: \mathbb{C}^d \to \mathbb{C}^d\) be a linear operator which is a contraction of all \(l_p\) spaces, simultaneously. Represent \(T\) by a matrix \((T_{ij})\) such that \((Tx)_i = \sum_j T_{ij} x_j, x = (x_i) \in \mathbb{C}^d\). Let \(H_{ij} = |T_{ij}|\) and let \(H: \mathbb{C}^d \to \mathbb{C}^d\) denote the corresponding operator, \((Hx)_i = \sum_j H_{ij} x_j\). Then it is easy to see that \(H\) is also a contraction of all \(l_p\) spaces and also that

\[(3.1) \quad r_j = \frac{1}{m_j} \sum_i H_{ij} m_i \leq 1,\]

\[(3.2) \quad s_i = \sum_j H_{ij} \leq 1.\]
These inequalities express the facts that $H$ is an $l_1$- $l_\infty$-contraction, respectively. Finally observe that

$$\sigma = \sum_j (1 - r_j) m_j = \sum_i (1 - s_i) m_i.$$  

(3.3)

The linear operators $T$ and $H$ will now be represented by an invertible measure preserving transformation $\tau$ of a $\sigma$-finite nonatomic measure space $(Z, \mu)$, as follows.

(3.4) DEFINITION OF $Z$. The subscripts $i$ and $j$ range through the integers $1, 2, \ldots, d$, as before. The superscript $k$ will range through the set of all integers $0, \pm 1, \pm 2, \ldots$. The measure space $(Z, \mu)$ will consist of a part of the $xy$-plane, together with the two dimensional Lebesgue measure. The length of an interval $I$ is denoted by $l(I)$.

Let $I^k_i$ be the disjoint intervals on the $x$-axis and $J^k_j$ disjoint intervals on the $y$-axis such that $l(I^k_i) = m_i$ and $l(J^k_j) = 1$. We then let

$$Z = \bigcup_k \bigcup_i I^k_i \times J^k_j.$$  

(3.5)

The points of $Z$ are denoted by $z$ or by $(x, y)$, as usual. A function $f$ defined on $Z$ is said to be constant on vertical lines if $f$ depends only on the $x$-coordinate of $(x, y) \in Z$. A subset $G$ of $Z$ is called a vertical set if its characteristic function $\chi_G$ is constant on vertical lines. Functions on $Z$ that are constant on horizontal lines and horizontal subsets of $Z$ are defined similarly. Note that each rectangle $I^k_i \times J^k_j$ is both a vertical and a horizontal subset of $Z$.

We will deal with partitions of $Z$ into finitely many atoms, which will contain atoms of infinite measure. The conditional expectation with respect to such a partition $\mathcal{G}$ will be denoted by $E(\cdot | \mathcal{G})$ and defined as zero on atoms of infinite measure. In particular $\mathcal{P}$ will denote the partition of $Z$ into $d + 2$ atoms, consisting of $d$ atoms $P_i = I^0_i \times J^0_j$ of finite measure and 2 atoms

$$Z^- = \bigcup_{k < 0} \bigcup_i I^k_i \times J^k_j, \quad Z^+ = \bigcup_{k > 0} \bigcup_i I^k_i \times J^k_j$$

of infinite measure. The condition expectation with respect to $\mathcal{P}$ will also be denoted by $E$. Note that there is a natural positive isometric isomorphism between $l_p$ and $EL_p(Z, \mu)$. We will not write this isomorphism explicitly and treat these two spaces as identical.

(3.6) AUTOMORPHISMS OF $Z$. All automorphisms of $Z$ considered in this paper are invertible and measure preserving transformation $\tau: Z \to Z$ of the following special type. Each rectangle $I^k_i \times J^k_j$ is partitioned into $2d$ horizontal rectangles $R^k_{ij}$ and $\hat{R}^k_{ij}$, some of them possibly of zero measure. Also, each rectangle $I^k_i \times J^k_j$ is partitioned into $2d$ vertical rectangles $S^k_{ij}$ and $\hat{S}^k_{ij}$ such that $\mu(R^k_{ij}) = \mu(S^k_{ij})$ and $\mu(\hat{R}^k_{ij}) = \mu(\hat{S}^k_{ij})$ and such that these measures depend only on $i$ and $j$, but not on $k$. Then $\tau$ maps $R^k_{ij}$ and $\hat{R}^k_{ij}$ onto $S^k_{ij}$ and $\hat{S}^{k+1}_{ij}$, respectively, and the restriction of $\tau$ to each $R^k_{ij}$ or $\hat{R}^k_{ij}$ is of the form

$$\tau(x, y) = (ax + b, cy + d),$$

where $a, b, c, d$ are four constants depending on $R^k_{ij}$ or $\hat{R}^k_{ij}$. 

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Let $Q$ be the induced operator on functions defined by
$$(Qf)(z) = f(\tau^{-1}z), \quad z \in \mathbb{Z}, \ f: \mathbb{Z} \to \mathbb{C}.$$ It is easy to see that one can always choose the measures of $R^k_{ij}$ and $\tilde{R}^k_{ij}$ in such a way that the operator $H$ is represented as
$$(3.7) \quad H = EQE.$$ In fact it is enough to take $\mu(R^k_{ij}) = \mu(S^k_{ij}) = H_{ij}m_i$, and define $\mu(\tilde{R}^k_{ij}) = \mu(\tilde{S}^k_{ij})$ rather arbitrarily, subject only to the requirements that
$$\sum_i (\mu(R^k_{ij}) + \mu(\tilde{R}^k_{ij})) = m_j \quad \text{and} \quad \sum_j (\mu(S^k_{ij}) + \mu(\tilde{S}^k_{ij})) = m_i.$$ Note that if $\sigma = 0$, as defined in (3.3), then we must take $\mu(R^k_{ij}) = \mu(S^k_{ij}) = 0$. Otherwise we may let, for example,
$$\mu(\tilde{R}^k_{ij}) = \mu(\tilde{S}^k_{ij}) = (1/\sigma)(1 - s_i)(1 - r_j)m_i m_j.$$ We now observe that if $Q$ represents $H$, as in (3.7), then we can also represent the original operator $T$ as
$$(3.8) \quad T = E\phiQE = EK$$ where $\phi: \mathbb{Z} \to \mathbb{C}$ is a function with constant unit modulus everywhere and $K = \phi Q$ denotes the operator on functions defined by $Kf = \phi(Qf)$. Furthermore one can choose $\phi$ as a function measurable with respect to the partition $\mathcal{P} \lor \tau \mathcal{P}$. In fact, we can define $\phi$ as follows.
$$\phi(z) = \begin{cases} \frac{T_{ij}}{H_{ij}} & \text{if } z \in S^0_{ij} \text{ and } \mu(S^0_{ij}) > 0, \\ 1 & \text{otherwise.} \end{cases}$$ Note that both $Q$ and $K$ are invertible isometries of all $L_p(\mathbb{Z}, \mu)$ spaces. Hence we see that $K^* = K^{-1}$.

(3.9) **LEMMA.** Let a function $f$ be constant on vertical lines and zero on $\mathbb{Z}^-$. Let $\mathcal{G}$ be a partition consisting of vertical atoms and refining $\mathcal{P}$. Then $E(f|\mathcal{G}) = E(f|\mathcal{G} \lor \tau^{-1}\mathcal{P})$.

**PROOF.** Both of these expectations are zero on $\mathbb{Z}^-$, since $\mathbb{Z}^-$ is a union of some atoms of $\mathcal{G}$ and $f = 0$ on $\mathbb{Z}^-$. Also, since $\tau^{-1} \mathbb{Z}^+ \supset \mathbb{Z}^+$ the restrictions of $\mathcal{G}$ and $\mathcal{G} \lor \tau^{-1}\mathcal{P}$ to $\mathbb{Z}^+$ are the same. On the remaining part of $\mathbb{Z}$ each atom $G$ of $\mathcal{G}$ has finite measure and is a vertical subset of one of the rectangles $P_j$. In $\mathcal{G} \lor \tau^{-1}\mathcal{P}$, this atom $G$ is further divided by the horizontal sets $R^0_{ij}$ and $\bigcup_i R^0_{ij}$. Since $f$ is constant on vertical lines, the average value of $f$ on each of these pieces is the same and is equal to the average value of $f$ on $G$. This shows that the two expectations in the lemma are the same on $G$.

(3.10) **LEMMA.** Let a function $f$ be constant on horizontal lines and zero on $\mathbb{Z}^+$. Let $\mathcal{G}$ be a partition consisting of horizontal atoms and refining $\mathcal{P}$. Then $E(f|\mathcal{G}) = E(f|\mathcal{G} \lor \tau \mathcal{P})$.

The proof is similar to the one above and is omitted.
(3.11) LEMMA. Let $f$ be constant on vertical lines and zero on $Z^-$. Let $\mathcal{F}$ be a vertical refinement of $\mathcal{P}$. Then

$$KE(f|\mathcal{F}) = E(Kf|\mathcal{P} \lor \tau \mathcal{F}).$$

PROOF. First note that, since $\tau$ is an invertible and measure preserving transformation,

$$QE(f|\mathcal{F}) = E(f|\mathcal{F}) \circ \tau^{-1} = E(f \circ \tau^{-1}|\tau \mathcal{F}) = E(Qf|\tau \mathcal{F}),$$

for any $f$ and $\mathcal{F}$. Under the hypotheses of the lemma we also have, by (3.9),

$$E(f|\mathcal{F}) = E(f|\mathcal{F} \lor \tau^{-1}\mathcal{P}).$$

Hence

$$KE(f|\mathcal{F}) = \phi QE(f|\mathcal{F} \lor \tau^{-1}\mathcal{P}) = \phi E(Qf|\mathcal{P} \lor \tau \mathcal{F}) = E(\phi Qf|\mathcal{P} \lor \tau \mathcal{F}) = E(Kf|\mathcal{P} \lor \tau \mathcal{F}),$$

where the third equality follows from the facts that $\phi$ is $\mathcal{P} \lor \tau \mathcal{P}$-measurable and that $\mathcal{P} \lor \tau \mathcal{F}$ is finer than $\mathcal{P} \lor \tau \mathcal{P}$.

(3.12) LEMMA. Let $f$ be constant on horizontal lines and zero on $Z^+$. Then

$$K^{-1}Ef = E(K^{-1}f|\mathcal{P} \lor \tau^{-1}\mathcal{P}).$$

PROOF. Since $Ef = E(f|\mathcal{P}) = E(f|\mathcal{P} \lor \tau \mathcal{P})$ by Lemma (3.10),

$$K^{-1}Ef = Q^{-1}\phi^{-1}Ef|\mathcal{P} \lor \tau \mathcal{P} = Q^{-1}E(\phi^{-1}f|\mathcal{P} \lor \tau \mathcal{P}) = E(K^{-1}f|\mathcal{P} \lor \tau^{-1}\mathcal{P}).$$

4. The main argument. Let $T_m$, $m = 1, \ldots, n$, be $n$ operators on a $d$-dimensional space, that contract all $l_p$-norms, $1 \leq p \leq \infty$, as in the previous section. Let $(Z, \mu)$ be the measure space, as constructed above, and let $\tau_m$ be the invertible measure preserving automorphisms of $Z$, associated with $T_m$, so that $T_m = EK_mE, K_m = \phi_m Q_m$, with the corresponding $L_p(Z, \mu)$-isometries $K_m$ and $Q_m$, as before.

(4.1) LEMMA. $T_m \cdots T_1Ef = EK_m \cdots K_1Ef$, for any $m = 1, \ldots, n$.

PROOF. The assertion is correct for $m = 1$. To prove the general case, first note that if a function $g$ satisfies the hypotheses of Lemma (3.11), then $K_m g$ also satisfies these hypotheses, for any $m$. Hence we see that $K_m \cdots K_1Ef$ satisfies these hypotheses. Then

$$K_{m+1}E(K_m \cdots K_1Ef) = E(K_{m+1}K_m \cdots K_1Ef|\mathcal{P} \lor \tau_{m+1}\mathcal{P}),$$

by Lemma (2.11). Since

$$EE(|\mathcal{P} \lor \tau_{m+1}\mathcal{P}) = E,$$

we have that

$$EK_{m+1}EK_m \cdots K_1Ef = EK_{m+1}K_m \cdots K_1Ef.$$
(4.2) LEMMA. $T^*_1 \cdots T^*_m E f = E K^{-1}_1 \cdots K^{-1}_m E f$ for any $m = 1, \ldots, n$.

We omit the proof as it is similar to the proof above.

(4.3) LEMMA. Starting with a function $f$ define a sequence $g_0, g_1, \ldots, g_n$ as

\begin{align*}
g_0 &= K_n \cdots K_1 E f, \\
g_m &= K_n \cdots K_{m+1} E K_m \cdots K_1 E f, \quad 1 \leq m < n, \\
g_n &= E K_n \cdots K_1 E f.
\end{align*}

Then there is a monotone sequence of partitions $\mathcal{G}_0 > \cdots > \mathcal{G}_n$ such that $g_m = E (g_0 | \mathcal{G}_m), 0 \leq m \leq n$.

PROOF. It is clear that $\mathcal{G}_n = \mathcal{P}$. If $1 \leq m < n$, let $h = K_m \cdots K_1 E f$. Then $h$ is a function that satisfies the hypotheses of Lemma (3.11). Hence we see that

\begin{align*}
g_m &= K_n \cdots K_{m+1} E h \\
&= K_n \cdots K_{m+2} E (K_{m+1} h | \mathcal{P} \vee \tau_{m+1} \mathcal{P}) \\
&= E (g_0 | \mathcal{G}_m),
\end{align*}

with

$\mathcal{G}_m = \mathcal{P} \vee \tau_n \mathcal{P} \vee \cdots \vee \tau_n \cdots \tau_{m+1} \mathcal{P}$,

by $(n - m)$ applications of Lemma (3.11).

(4.4) LEMMA. With the notations of the previous lemma,

\begin{equation}
\left\| \max_{0 \leq m \leq n} |g_m - g_n| \right\|_p \leq \frac{p}{p - 1} \|g_0 - g_n\|_p,
\end{equation}

and

\begin{equation}
\left\| \max_{0 \leq m \leq n} |g_m - g_0| \right\|_p \leq \frac{2p}{p - 1} \|g_0 - g_n\|_p,
\end{equation}

for any $1 < p < \infty$.

PROOF. The sequence $(g_m - g_n), m = 0, \ldots, n$, is a martingale, since

$g_m - g_n = E (g_0 - g_n | \mathcal{G}_m)$

and $\mathcal{G}_0 > \mathcal{G}_1 > \cdots > \mathcal{G}_n$. Hence the first inequality follows from the martingale maximal theorem. The second inequality is an obvious consequence of the first inequality.

(4.5) THEOREM. Given $p$, $1 < p < \infty$, and $\varepsilon > 0$, there is a $\delta = \delta(p, \varepsilon) > 0$, depending only on $p$ and $\varepsilon$, such that if

$\|E f\|_p - \|T_n \cdots T_1 E f\|_p < \delta \|E f\|_p$

then

$\left\| \max_{1 \leq m \leq n} |f_m - E f| \right\|_p < \varepsilon \|E f\|_p,$

with

$f_m = T^*_1 \cdots T^*_m T_m \cdots T_1 E f, \quad 1 \leq m \leq n.$

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PROOF. Let $V = K_1^{-1} \cdots K_n^{-1}$. Then we see that

\[
f_m = EK_1^{-1} \cdots K_m^{-1} EK_m \cdots K_1 Ef = EV K_n \cdots K_{m+1} EK_m \cdots K_1 Ef = EV g_m.
\]

Hence

\[
f_m - Ef = EV(g_m - g_0).
\]

Since $V$ is an invertible isometry of each $L_p(Z, \mu)$ space, $1 \leq p \leq \infty$, we see easily that

\[
\max_{1 \leq m \leq n} \left\| f_m - Ef \right\|_p \leq \max_{1 \leq m \leq n} \left\| g_m - g_0 \right\|_p \leq \frac{2p}{p-1} \left\| g_0 - g_n \right\|_p,
\]

where the second inequality follows from Lemma (4.4).

Now $g_n$ is the condition expectation of $g_0$ with respect to a partition $\mathcal{F}_n$. Hence, as observed in [3], for each $\xi > 0$, there is an $\eta > 0$, depending only on $\xi$ and $p$, such that if

\[
\left\| g_0 \right\|_p - \left\| g_n \right\|_p < \eta \left\| g_0 \right\|_p
\]

then

\[
\left\| g_0 - g_n \right\|_p < \xi \left\| g_0 \right\|_p.
\]

But

\[
\left\| g_0 \right\|_p = \left\| K_n \cdots K_1 Ef \right\|_p = \left\| Ef \right\|_p
\]

and

\[
\left\| g_n \right\|_p = \left\| EK_n \cdots K_1 Ef \right\|_p = \left\| T_n \cdots T_1 Ef \right\|_p.
\]

Hence it is enough to choose $\xi > 0$ sufficiently small so that

\[
\frac{2p}{p-1} \xi < \varepsilon
\]

and then take $\delta$ to be equal to the corresponding $\eta > 0$.

An observed earlier, the main result, Theorem (2.1), follows easily from the result above.

REFERENCES