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ON FUNCTIONS THAT ARE TRIVIAL COCYCLES FOR A SET OF IRRATIONALS

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ABSTRACT. The main result of this paper is that the set of irrationals, for which a given function is a trivial cocycle, must be of the first category, unless the function is the exponential of a trigonometric polynomial.

DEFINITION. Let θ be a fixed irrational number. A measurable function $f\colon [0,1) \to \mathbb{T}$ is called a *Trivial multiplicative cocycle* for θ if there exists a scalar λ and a function $g : [0,1) \to \mathbb{T}$ such that

$$f(x) = \lambda g(x)/g(x+\theta).$$

For various reasons, this author and others (See [**1**, **2**, and **3**].) have been interested in when a given function can be simultaneously a trivial cocycle for two irrationals θ_1 and θ_2, or, for that matter, a trivial cocycle simultaneously for an entire family of θ's. If g is any measurable function from $[0,1)$ to $\mathbf{T}$, then $f(x) = g(x)/g(x+\theta)$ defines a trivial cocycle for θ. If ϕ is a second irrational number, then $h(x) = f(x)/f(x+\phi)$ defines a trivial cocycle for ϕ which is also a trivial cocycle for θ. Obviously, this method can be used to construct functions that are simultaneously trivial cocycles for any finite number of irrationals.

In [**2**], it is shown that if $f = e^{2\pi i v}$, for v real-valued, periodic, absolutely continuous, and with L^2 derivative, then f is a trivial cocycle for every badly approximable irrational θ. Badly approximable irrationals are precisely the ones having bounded partial quotients in their continued fraction expansions. These numbers constitute an uncountable set of measure zero. The theorems below shed some more light on the size of a class of irrationals for which a function can simultaneously be a trivial cocycle.

THEOREM 1. *If f is the exponential of a real-valued trigonometric polynomial, i.e., if*

$$f(x) = \exp\left(2\pi i \sum_{n=0}^{N} a_n \cos(2\pi n x) + b_n \sin(2\pi n x)\right),$$

then f is a trivial cocycle for every irrational number θ.

PROOF. This argument is perhaps well-known to experts in the subject of small divisors. If θ is any irrational number, n is any nonzero integer, and $w\colon \mathbf{R} \to \mathbf{R}$ is defined by

$$w(x) = \frac{e^{2\pi i n x}}{1 - e^{2\pi i n\theta}},$$

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Then clearly

$$e^{2\pi inx} = w(x) - w(x+\theta),$$

which shows that any trigonometric polynomial is a trivial additive cocycle for any θ. The theorem now follows by exponentiating.

THEOREM 2. *Suppose $f\colon \mathbf{R} \to \mathbf{T}$ is periodic with period 1 and is continuous on the half-open interval $[0,1)$. Let S be a subset of $\mathbf{R}$ which is not of the first category, i.e., is not contained in a union of countably many closed nowhere dense sets. Then f is a trivial multiplicative cocycle for every irrational number θ in S if and only if f is the exponential of a real-valued trigonometric polynomial.*

PROOF. Write $f(x) = e^{2\pi iv(x)}$ for $v\colon [0,1) \to \mathbf{R}$ continuous on $[0,1)$, and, for each irrational number θ in S, let g_θ and λ_θ be such that

$$f(x) = \lambda_\theta g_\theta(x)/g_\theta(x+\theta).$$

For each integer j and each positive integer k, let $A_{j,k}$ be the set of all real numbers θ for which there exists a function $g_\theta\colon [0,1) \to \mathbf{T}$ and a constant λ_θ such that

$$f(x) = \lambda_\theta g_\theta(x)/g_\theta(x+\theta) \tag{1}$$

and

$$\left|\int_0^1 g_\theta(x)e^{2\pi ijx}dx\right| \geq \frac{1}{k}. \tag{2}$$

The assumptions of the theorem imply that every irrational number in S belongs to some $A_{j,k}$. We claim that each set $A_{j,k}$ is closed in $\mathbf{R}$. Thus, let $\{\theta_n\}$ be a sequence of elements of $A_{j,k}$ which converges to a number θ. From the definition of $A_{j,k}$, we see that the sequence $\{g_{\theta_n}\}$ does not converge weakly to 0 in L^2, and we may assume, by passing to a subsequence if necessary, that $\{g_{\theta_n}\}$ converges weakly to a nonzero function h, and that the corresponding sequence $\{\lambda_{\theta_n}\}$ converges to a number λ.

For each $\phi \in L^2$, we have that

$$\begin{aligned}
\int f(x)h(x+\theta)\phi(x)dx &= \int h(x)f(x-\theta)\phi(x-\theta)dx \\
&= \lim \int g_{\theta_n}(x)f(x-\theta)\phi(x-\theta)dx \\
&= \lim \int g_{\theta_n}(x)f(x-\theta_n)\phi(x-\theta_n)dx \\
&= \lim \int f(x)g_{\theta_n}(x+\theta_n)\phi(x)dx \\
&= \lim \int \lambda_{\theta_n} g_{\theta_n}(x)\phi(x)dx \\
&= \lambda \int h(x)\phi(x)dx,
\end{aligned}$$

which shows that

$$f(x) = \lambda h(x)/h(x+\theta).$$

It follows from this that h has constant nonzero absolute value $m \leq 1$, and we define $g_\theta = (1/m)h$. Clearly, (1) is satisfied:

$$f(x) = \lambda g_\theta(x)/g_\theta(x+\theta).$$

Also, we have (2):

$$\left|\int g_\theta(x)e^{2\pi ijx}dx\right| \geq \left|\int h(x)e^{2\pi ijx}dx\right| \geq \frac{1}{k},$$

proving that $\theta \in A_{j,k}$, as desired.

By assumption, it follows that some $A_{j,k}$ has nonempty interior, which implies that there exists a positive integer N, such that for all $q > N$, $A_{j,k}$ contains a nonzero rational number $r = p/q$. For such an r then, there exists a function g and a scalar λ such that

$$f(x) = \lambda g(x)/g(x+r).$$

It follows that

$$\begin{aligned} f(x)f(x+r)\cdots f(x+(q-1)r) &= \lambda^q g(x)/g(x+qr) \\ &= \lambda^q g(x)/g(x+p) \\ &= \lambda^q. \end{aligned}$$

Hence,

$$e^{2\pi i(v(x)+v(x+r)+\cdots+v(x+(q-1)r))} = \lambda^q.$$

Since v is continuous on the half-open interval $[0,1)$, this implies that

$$v(x) + v(x+r) + \cdots + v(x+(q-1)r) = c + M(x), \tag{3}$$

for c a real constant and M an integer-valued function having the same points of discontinuity as the function on the lefthand side of equation (3). These points of discontinuity are only at certain multiples of $1/q$, whence the function M is a step function with jump discontinuities only at certain multiples of $1/q$. Computing Fourier coefficients of both sides of (3) gives that $qc_n(v) = 0$ for any n which is a multiple of q, and this implies that v is a real-valued trigonometric polynomial, which completes the "only if" part of the proof. The "if" part follows from Theorem 1.

REMARK. It follows from Theorems 1 and 2 that a continuous function $f\colon [0,1) \to \mathbb{T}$ is a trivial cocycle for every irrational θ if and only if f is the exponential of a real-valued trigonometric polynomial.

REMARK. Theorem 2 can be used to deduce that certain subsets of $\mathbb{R}$ are of the first category. For instance, if $\{a_n\}$ are the Fourier coefficients of a continuous real-valued function v on $[0,1)$, which is not a trigonometric polynomial, and if S is the set of all $\theta \in \mathbb{R}$ for which the series

$$\sum \frac{a_n}{1-e^{2\pi in\theta}}e^{2\pi inx}$$

is Abel summable to a function $f_\theta(x)$, then S must be of the first category.

As usual, the size of a set can be thought of topologically (as in the sense of category) or measure-theoretically. Here, as in other places, the two notions of size are dramatically different.

THEOREM 3. *There exists a continuous function $f\colon [0,1) \to \mathbb{T}$ which is a trivial cocycle for almost every irrational θ, but which is not the exponential of any real-valued trigonometric polynomial.*

PROOF. This argument is also one that might be familiar to experts. Let L denote the set of *Liouville* numbers. By definition, an irrational number θ is a

Liouville number if for every positive integer d, there exists a rational number m/n such that

$$|\theta - m/n| \leq 1/n^d.$$

It is known that the set of Liouville numbers is of Lebesgue measure 0.

Let S denote the complement of L. For each positive integer d, let S_d be the set of all θ for which there exists a positive constant $a_{\theta,d}$ such that

$$|\theta - m/n| \geq a_{\theta,d}/n^d$$

for every rational m/n. Then $S = \bigcup_d S_d$.

Now, set

$$v(x) = \sum 2^{-|n|} e^{2\pi i n x},$$

and

$$f(x) = e^{2\pi i v(x)}.$$

If θ belongs to S, choose d such that $\theta \in S_d$, and for each n let m be chosen so that $|\theta - m/n| \leq 1/2n$. We recall that if $|x| \leq 1/2$, then $|1 - e^{2\pi i x}| \geq 4|x|$. Hence,

$$|1 - e^{2\pi i n\theta}| = |1 - e^{2\pi i(m - n\theta)}| \geq 4|m - n\theta| \geq 4a_{\theta,d}/n^{d-1}.$$

It follows that the sequence $\{c_n(\theta)\}$, defined by

$$c_n(\theta) = \frac{1}{2^{|n|}(1 - e^{2\pi i n\theta})},$$

is absolutely summable, so that the function

$$w_\theta(x) = \sum_{n=-\infty}^{\infty} c_n(\theta) e^{2\pi i n x}$$

is continuous and periodic. Also, by comparing Fourier coefficients, we have that

$$v(x) = c_0(v) + w_\theta(x) - w_\theta(x + \theta).$$

Clearly, then, the function f is a trivial cocycle for every $\theta \in S$, and this completes the proof.

REMARK. It follows from Theorem 2 that the set S of the preceding proof is of the first category, i.e., that the set L of Liouville numbers is co-meager, which is another well-known fact about the Liouville numbers.

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