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AN IRREDUCIBLE NOT ADMISSIBLE BANACH REPRESENTATION OF $SL(2,\mathbf{R})$

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ABSTRACT. This weird representation can be constructed by inducing from a solution of the invariant subspace problem.

Introduction. Let us remark first that the real work has been done by P. Enflo [**1**] and C. J. Read [**2**, **3**], who constructed solutions (E,ψ) of the invariant subspace problem, i.e. a complex Banach space E of dimension greater then one and a bounded linear $\psi : E \to E$ such that 0 and E are the only ψ-invariant closed linear subspaces of E. Thus $t \mapsto \exp(t\psi)$ is an irreducible not admissible representation of $\mathbf{R}$, a reductive Lie group, in E.

Let us remark second that the real idea is due to David Vogan, namely to somehow induce this representation.

In more detail one may proceed as follows: Note first that ψ and $a1 + b\psi$ have the same invariant subspaces, for all $a \in \mathbb{C}$, $b \in \mathbb{C}^*$. Consider now $\phi = \sqrt{1 + (2\|\psi\|)^{-1}\psi} + 1$. Then $(E, (\phi - 1)^2)$ solves the invariant subspace problem.

Let's take the standard Iwasawa decomposition $SL(2,\mathbf{R}) = KAN$ and induce the representation

$$\kappa : \begin{pmatrix} \pm e^t & * \\ 0 & \pm e^{-t} \end{pmatrix} \mapsto \exp(t\phi)$$

of $P = MAN$ in E to a representation of $G = SL(2,\mathbf{R})$. I'll show this induced representation to be irreducible but not admissible.

Verification. For now let E be an arbitrary Banach space and $\phi\colon E \to E$ an arbitrary bounded linear map. We then define a representation κ of P in E as above and construct a Banach representation $\pi = \operatorname{ind}_P^G \kappa$ of G setting

$$\mathcal{H} = \{f\colon G \to E \mid f \text{ continuous}, f(gp) = \kappa(p)^{-1} f(g)\ \forall g \in G, p \in P\}.$$

With the norm $\|f\| = \sup_{k\in K} \|f(k)\|$ this is a Banach space and G acts continuously by $(\pi(g)f)(g_1) = f(g^{-1}g_1)$. Restriction of functions defines an isomorphism $\mathcal{H} \cong \{f\colon K \to E | f \text{ continuous}, f(-k) = f(k)\ \forall k \in K\}$. We refer to this as the "compact picture".

We investigate the K-finite vectors of $\mathcal{H}$. Let $S^1 = \{z \in \mathbb{C} |\ |z| = 1\}$ and define $k\colon S^1 \xrightarrow{\sim} K$ by $z = x + iy \mapsto \left(\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right)$. For $\nu \in \mathbb{Z}$ let $\mathcal{H}_\nu = \{f \in \mathcal{H} \mid \pi(k(z))f = z^\nu f\ \forall z \in S^1\}$ be the vectors of K-type ν. Via the compact picture we easily verify

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CLAIM 1. *For ν odd, $\mathcal{H}_\nu = 0$. For ν even, the map $f \mapsto f(k(1))$ gives an isomorphism of Banach spaces $\mathcal{H}_\nu \to E$. The inverse is given by $e \mapsto f(\nu, e)$ where $f(\nu, e)(k(z)) = z^{-\nu}e$.*

We need to establish and calculate the action of $\mathbf{g} = \mathrm{Lie}(G)_{\mathbf{C}}$ on our K-finite vectors. This is done by the following two claims whose proofs we postpone.

CLAIM 2. *For all C^∞-maps $\sigma\colon (-\varepsilon, +\varepsilon) \to G$ and for all $e \in E$, $\nu \in 2\mathbf{Z}$, the mapping $(-\varepsilon, +\varepsilon) \to \mathcal{H}$, $s \mapsto \pi(\sigma(s))f(\nu, e)$ is differentiable at $s = 0$.*

Consider in $\mathbf{g}$ the standard basis

$$X = \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, \quad H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Y = \frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}.$$

CLAIM 3.

$$\begin{aligned} \pi(X)f(\nu, e) &= f\left(\nu - 2, \tfrac{1}{2i}(\phi(e) - \nu e)\right), \\ \pi(Y)f(\nu, e) &= f\left(\nu + 2, -\tfrac{1}{2i}(\phi(e) + \nu e)\right), \\ \pi(H)f(\nu, e) &= f(\nu, -\nu e) \end{aligned}$$

It readily follows that $\pi(4XY + H^2 - 2H + 1)f(\nu, e) = f(\nu, (\phi - 1)^2 e)$. In particular for $(E, \phi) = (\mathbf{C}, \lambda)$ we obtain the usual formulas for the Harish-Chandra module of a principal series representation and its central character.

But now I choose (E, ϕ) as in the introduction, i.e. such that $(E, (\phi - 1)^2)$ solves the invariant subspace problem. I claim, that then $\mathcal{H}$ is an irreducible G-module (which clearly is not admissible). In fact, let $V \subset \mathcal{H}$ be a closed, nonzero, G-invariant subspace. Since the K-finite vectors of V are dense in V, we have $V \cap \mathcal{H}_\nu \neq 0$ for some ν. Clearly for $v \in V \cap \mathcal{H}_\nu$ and $Z \in \mathbf{g}$ we have $\pi(Z)v \in V$. So under the identification $E \cong \mathcal{H}_\nu$ the subspace $V \cap \mathcal{H}_\nu$ must correspond to a $(\phi - 1)^2$-invariant closed subspace of E, so $V \supset \mathcal{H}_\nu$, and further using the formulas of Claim 3, we finally get $V \supset \mathcal{H}_\nu$ for all ν. And since the K-finite vectors are dense in $\mathcal{H}$, this proves $V = \mathcal{H}$.

Completion. We now complete the paper by proving the Claims 2 and 3. Let σ be as in Claim 2. Fix $\nu \in 2\mathbf{Z}$, $e \in E$ and set $f = f(\nu, e)$.

Clearly $F\colon (t, k) \mapsto (\pi(\sigma(t))f)(k)$ is a C^∞-function $(-\varepsilon, +\varepsilon) \times K \to E$. I consider $\partial_1 F(0, k)$ as an element of $\mathcal{H}$ via the compact picture and claim that it is the derivative at 0 of $t \mapsto \pi(\sigma(t))f$. For this we but have to show that

$$\lim_{t \to 0}\left(\sup_{k \in K} \|F(t, k) - F(0, k) - t\partial_1 F(0, k)\|\right) = 0.$$

Put it otherwise:

$$\lim_{t \to 0}\left(\sup_{k \in K} \left\|\int_0^t (\partial_1 F(t', k) - \partial_1 F(0, k))dt'\right\|\right) = 0.$$

But this is evident by uniform continuity of $\partial_1 F$ on $[-\varepsilon/2, +\varepsilon/2] \times K$. So we have shown Claim 2 in so explicit a way that the verification of Claim 3 reduces to an elaborate computation. This computation is practically the same one usually does in the case of the principal series, so we can safely omit it.

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