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NONHOMOGENEITY OF CERTAIN BAIRE SETS IN βX

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ABSTRACT. We prove that every nonempty Baire set of βX contained in $\beta X - X$ is not homogeneous.

All spaces are assumed to be completely regular Hausdorff. Let X be a space. The family of Baire sets in X and the family of zero sets in X are denoted by $\mathscr{B}(X)$ and $\mathscr{Z}(X)$, respectively.

One of the most interesting theorems about nonhomogeneity is due to Z. Frolík [**F**]: $\beta X - X$ is not homogeneous whenever X is not pseudocompact. We shall use a generalization of this theorem, due to E. K. van Douwen [**vD**]. Before stating it, we give a definition.

DEFINITION. Let X be a space. A subset $S \subset X$ is shy if for every $A \subseteq S$ and for every countable discrete subset $B \subseteq X$ we have that $\overline{A} \cap \overline{B} = \varnothing$ whenever $\overline{A} \cap B = A \cap \overline{B} = \varnothing$.

LEMMA [**vD**]. *A space is non-homogeneous if it has a countably infinite discrete shy subset whose closure is homeomorphic to βN.*

The next lemma is a generalization of Lemma 2 of [**vD**]. Since the proof from [**vD**] suffices here, we shall not repeat it.

LEMMA. *If $G \subseteq \beta X - X$ is a nonempty G_δ-subset of βX, then G contains a countably infinite discrete shy subset of βX whose closure is homeomorphic to βN.*

As a direct consequence of the lemmas above we have:

THEOREM. *If $A \subseteq \beta X$ and there is a nonempty $Z \in \mathscr{Z}(\beta X)$ with $Z \subseteq A \cap \beta X - X$, then A is not homogeneous.*

Let X be a space. It is known that every Baire set in X is a union of zero sets of X. The referee notes that this property of Baire sets is exactly the Lemma 3.7 of [**N**]. For a direct proof, notice that $\mathscr{B}(X) = \bigcup\{Z_\xi : \xi < \omega_1\}$, where $\mathscr{Z}_0 = \mathscr{Z}(X)$, and

$$\begin{aligned} \mathscr{Z}_\xi &= \left\{ \bigcup_{n<\omega} B_n : B_n \in \bigcup_{\nu<\xi} \mathscr{Z}_\nu, n < \omega \right\} \quad \text{for odd } \xi < \omega_1, \\ &= \left\{ \bigcap_{n<\omega} B_n : B_n \in \bigcup_{\nu<\xi} \mathscr{Z}_\nu, n < \omega \right\} \quad \text{for even } \xi < \omega_1. \end{aligned}$$

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By transfinite induction we can prove that every Baire set in $\mathscr{Z}_\xi$, $\xi < \omega_1$, is a union of zero sets of X. Alternately, one may use the familiar result (see Lemma 9.1 in [**CN**]) that for every $A \in \mathscr{B}(X)$ there is a metrizable space M and a continuous function $f\colon X \to M$ such that $A = f^{-1}(f(A))$. Clearly the equation $A = \bigcup\{f^{-1}(\{p\})\colon p \in f(A)\}$ expresses A in the desired form. It is evident that this property of Baire sets and our theorem imply:

COROLLARY. *Every nonempty Baire set of βX contained in $\beta X - X$ is not homogeneous.*

REMARK. W. W. Comfort pointed out to me that this Corollary follows from the result of Frolík quoted above. Indeed, let A be a nonempty Baire set of βX contained in $\beta X - X$. Set $Y = \beta X - A$. Then $Y \in \mathscr{B}(\beta X)$; hence Y is a Lindelöf space since every Baire set of a compact space is Lindelöf (see 9.10(a) in [**CN**] for a proof). Hence, Y is not pseudocompact since $X \subsetneqq Y \subsetneqq \beta X$. Therefore, $A = \beta Y - Y$ is non-homogeneous by Frolík's theorem.

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REFERENCES

[**CN**] W. W. Comfort and S. Negrepontis, *Continuous pseudometrics*, Lecture Notes in Pure and Appl. Math., No. 14, Marcel Dekker, New York, 1975.

[**vD**] E. K. van Douwen, *Homogeneity of βG if G is a topological group*, Colloq. Math. **41** (1979), 193–199.

[**F**] Z. Frolík, *Nonhomogeneity of $\beta X - X$*, Comm. Math. Univ. Carolin., **8** (1967), 705–709.

[**N**] S. Negrepontis, *Baire sets in topological spaces*, Arch. Math., **18** (1967), 603–608.

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