AN INEQUALITY FOR SOME NONNORMAL OPERATORS

TAKAYUKI FURUTA

(Communicated by Daniel W. Stroock)

Dedicated to Professor Tsuneo Kanno on his sixtieth birthday
with respect and affection

ABSTRACT. An inequality of use in testing convergence of eigenvector calculations is improved. If $e_{\lambda}$ is a unit eigenvector corresponding to an eigenvalue $\lambda$ of a dominant operator $A$ on a Hilbert space $H$, then

$$|(g, e_{\lambda})|^2 \leq \frac{\|g\|^2 \|Ag\|^2 - |(g, Ag)|^2}{\|(A - \lambda I)g\|^2}$$

for all $g$ in $H$ for which $Ag \neq \lambda g$. The equality holds if and only if the component of $g$ orthogonal to $e_{\lambda}$ is also an eigenvector of $A$. This result is an improvement of Bernstein’s result for selfadjoint operators.

1. Statement of the results. An operator $A$ means a bounded linear operator on a complex Hilbert space $H$. An operator $A$ is called dominant if there is a real number $M_\lambda \geq 1$ such that

$$\|(A - \lambda)^* x\| \leq M_\lambda \|(A - \lambda) x\|$$

for all $x$ in $H$ and for all complex numbers $\lambda$. If there is a constant $M$ such that $M_\lambda \leq M$ for all $\lambda$, $A$ is called $M$-hyponormal. The inclusion relation of these classes of nonnormal operators is as follows:

$$\text{Selfadjoint} \subset \text{Normal} \subset \text{Quasinormal} \subset \text{Subnormal} \subset \text{Hyponormal}$$

$$\subset M\text{-hyponormal} \subset \text{Dominant},$$

and it is well known that the inclusions above are all proper.

THEOREM 1. If $e_{\lambda}$ is a unit eigenvector corresponding to an eigenvalue $\lambda$ of a dominant operator $A$ on a Hilbert space $H$, then

$$|(g, e_{\lambda})|^2 \leq \frac{\|g\|^2 \|Ag\|^2 - |(g, Ag)|^2}{\|(A - \lambda)g\|^2}$$

for all $g$ in $H$ for which $Ag \neq \lambda g$. The equality holds if and only if the component of $g$ orthogonal to $e_{\lambda}$ is also an eigenvector of $A$. The bound of the right-hand side is $\|(A - \tau)g\|^2/|\lambda - \tau|^2$ for any complex $\tau$.

The corresponding result for selfadjoint operators in Theorem 1 is shown by Bernstein [1]. In this note, we extend the Bernstein result to the class of dominant operators, wider than the one of selfadjoint operators, by appropriate modification of Bernstein [1] together with the following lemma.

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LEMMA 1. Let $H$ be a Hilbert space, and $A$ any operator on $H$. Then

(i) $\|x\|^2\|Ax\|^2 - |(x, Ax)|^2 = \|x\|^2\|(A - \tau)x\|^2 - |(x, (A - \tau)x)|^2$

for any vector $x$ in $H$ and any complex $\tau$.

(ii) $\|x\|^2\|Ax\|^2 - |(x, Ax)|^2 \leq \|(A - \lambda)x\|^2 / |\lambda - \tau|^2$

for all $x$ in $H$ for which $Ax \neq \lambda x$ and all complex $\tau$ and $\lambda$.

PROOF OF LEMMA 1. (i) is already shown in [2] by straightforward calculation and (ii) is also easily obtained by using (i).

LEMMA 2. Let $H$ be a Hilbert space, $A$ a dominant operator on $H$, $e_\lambda$ a unit eigenvector of $A$ with corresponding eigenvalue $\lambda$, and let $f$ be orthogonal to $e_\lambda$. Then for any $g = \alpha e_\lambda + f$ with $\alpha$ complex, either $Ag = \lambda g$ or

$$|\alpha|^2 \leq \|g\|^2\|Ag\|^2 - |(g, Ag)|^2 / \|(A - \lambda)g\|^2$$

PROOF OF LEMMA 2. First of all, we have $(A - \lambda)g = (A - \lambda)f$ and

$$(g, (A - \lambda)g) = (g, (A - \lambda)f) = (\alpha e_\lambda, (A - \lambda)f) + (f, (A - \lambda)f)$$

$$= \alpha((A - \lambda)^* e_\lambda, f) + (f, (A - \lambda)f)$$

$$= (f, (A - \lambda)f)$$

because the hypothesis $(A - \lambda)e_\lambda = 0$ yields $(A - \lambda)^* e_\lambda = 0$ by the definition of dominant operator $A$. Put $B = A - \lambda$. Since $Bg = Bf$ and $(g, Bg) = (f, Bf)$, by Lemma 1 and the Schwarz inequality, we have

$$\|g\|^2\|Ag\|^2 - |(g, Ag)|^2 / \|(A - \lambda)g\|^2 = (|\alpha|^2 + \|f\|^2)\|Bf\|^2 - |(f, Bf)|^2 / \|Bf\|^2$$

$$\geq |\alpha|^2.$$

PROOF OF THEOREM 1. Inequality is shown by Lemma 1 and Lemma 2, and it is also seen that the equality holds if and only if $f$ is an eigenvector of $B$, equivalently, $f$ is an eigenvector of $A$ in the proof of Lemma 2.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, HIROSAKI UNIVERSITY, BUNKYO-CHO 3, HIROSAKI, 036 AOMORI, JAPAN