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ON THE PERIODIC POINTS OF A TYPICAL CONTINUOUS FUNCTION

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ABSTRACT. Let n and k be arbitrary natural numbers. We prove that for a typical continuous function f, every neighborhood of any periodic point of f with period n contains periodic points of f with period $n \cdot k$.

The purpose of this paper is to examine the structure of the set of periodic points of typical continuous functions. We shall denote by C the set of continuous functions mapping the interval $[0, 1]$ into itself. This set becomes a metric space with the supremum metric. By the term "typical continuous function" we mean that the set of all functions having the property under consideration is a residual subset of the metric space C.

In [1] S. J. Agronsky, A. M. Bruckner, and M. Laczkovich proved that for a typical continuous function f any neighborhood of a periodic point contains periodic points of arbitrarily large periods. In this paper we prove that this result is true if "arbitrarily large periods" is replaced by "period $k \cdot n$ $(k = 1, 2, \ldots)$" (Theorem 1). As a consequence, for a typical continuous f we have that P_f^n, the set of periodic points with period n, is uncountable, dense in itself, but not closed. Also, P_f^n is a residual subset of $\mathrm{Fix}(f^n)$.

In [1] it is proved that, for a typical continuous f, $\mathrm{Fix}(f^n)$ is nowhere dense and perfect. In this paper we prove (Theorem 2) that $\mathrm{Fix}(f^n)$ is bilaterally strongly Φ-porous. For $f \in C$ and $n \in \mathbf{N}$ we define $f^n(x)$ by $f^1(x) = f(x)$ and $f^n(x) = f(f^{n-1}(x))$. P_f^n denotes the set of periodic points of f with period n. We denote $\mathrm{Fix}(f^n) = \{x : f^n(x) = x\}$; then $P_f^n = \mathrm{Fix}(f^n) \setminus \bigcup_{k<n} \mathrm{Fix}(f^k)$. In particular P_f^n is a relatively open subset of $\mathrm{Fix}(f^n)$. For $f \in C$ and $\varepsilon > 0$ denote

$$B(f, \varepsilon) = \{g \in C : \|f - g\| < \varepsilon\}.$$

Theorem 1. *Let $n, k \in \mathbf{N}$ be arbitrary. Then for a typical continuous function f, every neighborhood of any periodic point of f with period n contains periodic points of f with period $n \cdot k$.*

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Proof. Fix $n \in \mathbf{N}$. Let $A_0 = \{f \in C\colon \operatorname{Fix}(f^n) \text{ nowhere dense}\}$ and $A_i = \{f \in C\colon$ every neighborhood of any periodic point of f with period i contains, for every $m \in \mathbf{N}$, periodic points of f with period $m \cdot i\}$ $(i = 1, 2, \dots)$.

We prove by induction that each A_i is residual. In [1] it is proved that the set A_0 is a residual subset of C. Assume that $\bigcap_{0 \le i < n} A_i$ is a residual subset of C. We shall prove that A_n is also residual. If $f \in \bigcap_{0 \le i < n} A_i$ and I is an arbitrary open interval then $P_f^n \cap I = \varnothing$ implies $\operatorname{Fix}(f^n) \cap I = \varnothing$. For $k \in \mathbf{N}$ and $\varepsilon > 0$ we define the set of Q_k^ε by

$$Q_k^\varepsilon = \{f \in C\colon \text{ for any } x_0 \in P_f^n \text{ there is } x \in P_f^{k \cdot n} \text{ such that } |x - x_0| < \varepsilon\}.$$

It will be sufficient to show that each Q_k^ε contains a set which is open and dense. Let $f \in \bigcap_{0 \le i < n} A_i$ and let $\varepsilon' > 0$ be arbitrary. We prove that there exist $g \in B(f, \varepsilon')$ and $\eta > 0$ such that $B(g, \eta) \subset Q_k^\varepsilon$. Let $\{a_i\}_{i=0}^l \subset [0, 1] \setminus \operatorname{Fix}(f^n)$ be a finite sequence such that $a_0 = 0$, $a_l = 1$ and $0 < a_{i+1} - a_i < \varepsilon$ for every $0 \le i < l$. Put $J_i = [a_{i-1}, a_i]$ $(i = 1, \dots, l)$. Choose a finite sequence $\{b_j\}_{j=1}^z \subset P_f^n$ as follows: if $J_i \cap P_f^n \ne \varnothing$ $(i = 1, \dots, l)$ then there should exist an $x_0 \in J_i \cap P_f^n$ such that the sequence $\{b_j\}_{j=1}^z$ contains the orbit of x_0. Let $z' = z/n$. We can choose $b_1, b_2, \dots, b_z$ so that they are contained in different orbits. Choose $\eta_1 \in \mathbf{R}$ so that

(i) $\eta_1 > 0$,

(ii) $\eta_1 < \frac{1}{3} \cdot \min\limits_{\substack{0 \le i_1 < i_2 \le l \\ 0 \le j_1 < j_2 \le z \\ 0 \le i \le l,\ 0 \le j \le z}} \{|a_{i_1} - a_{i_2}|, |b_{j_1} - b_{j_2}|, |a_i - b_j|\}$,

(iii) $\eta_1 < \varepsilon'/3$, and

(iv) if for some i $J_i \cap P_f^n = \varnothing$ and $\|g - f\| < 3\eta_1$ then $J_i \cap P_g^n = \varnothing$.

This is possible since $f \in \bigcap_{i<n} A_i$. Let $0 < u < \eta_1$ be such that if $|x - y| < u$ then $|f(x) - f(y)| < \eta_1$. Then $[b_i - u/2, b_i + u/2] \cap [b_j - u/2, b_j + u/2] = \varnothing$ for every $1 \le i < j \le z$.

We can now define the function $g \in C$. First we define g on

$$\bigcup_{i=1}^n [f^i(b_m) - u/2, f^i(b_m) + u/2] \qquad (m = 1, \dots, z')$$

using the following procedure. Let $1 \le m \le z'$ be fixed but arbitrary and take $x_0 = b_m$. Put $I_j = [f^{j-1}(x_0) - u/2, f^{j-1}(x_0) + u/2]$. Let us define g on the set $\bigcup_{j=1}^n I_j$ so that the graph of g on this set will be included in the squares $E_j = I_j \times I_{j+1}$ (see Figure 1). If $j > 1$ then let the graph of g in E_j be the increasing diagonal of E_j. (See Figure 1.) Let $x_1, \dots, x_k$, $x_{k+1} = x_1$ be distinct points in $(0, u)$ and let the point W_i be defined by $W_i = (x_i, x_{i+1})$ $(i = 1, \dots, k)$; then $W_i \in L = [0, u] \times [0, u]$. Let $v = \min_{i \ne j} \frac{1}{5}|x_i - x_j|$ and define

$$B_i = \left[x_i - \tfrac{v}{2}, x_i + \tfrac{v}{2}\right] \times \left[x_{i+1} - \tfrac{v}{2}, x_{i+1} + \tfrac{v}{2}\right].$$

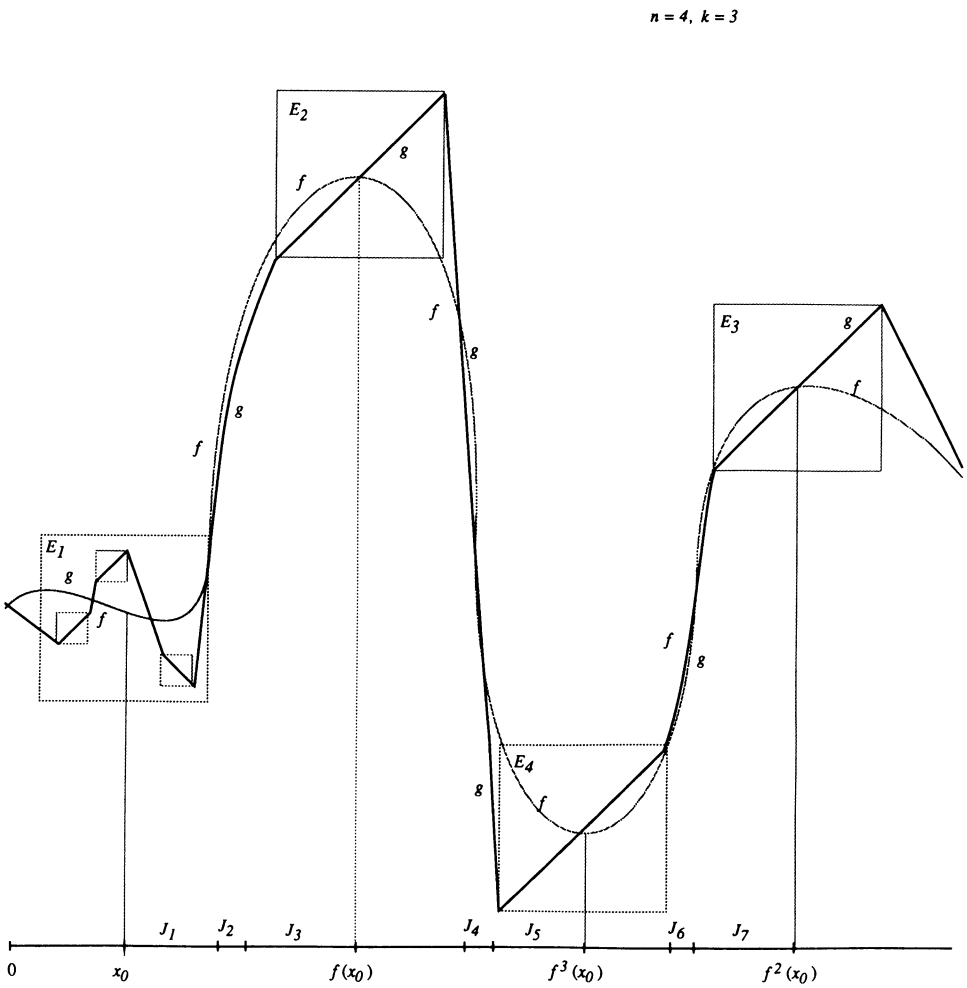

FIGURE 1

Let $h\colon [0,u] \to [0,u]$ be piecewise linear and continuous such that the graph of h in each B_i is the diagonal of B_i, if $1 \le i < k$ then h is increasing in B_i but h is decreasing in B_k. (See Figure 2.)

It is easy to see that if $\eta = v/10$ and $q \in B(h,\eta)$ then q has periodic points of period k in the interval $(x_i - v/2, x_i + v/2)$. Now we define the graph of g on I_1 by translating the graph of h into E_1 (using the translation that maps L onto E_1). Thus we have defined g on $\bigcup_{i=1}^{z}[b_i - u/2, b_i + u/2]$. Put $g(a_i) = f(a_i)$ $(i = 0, 1, \dots, l)$. If $J_i \cap P_f^n = \varnothing$ then let $g|_{J_i} = f|_{J_i}$. For those x's for which g has not yet been defined, let us define g so that $\|f - g\| < \eta_1$; by the definition of u this is possible. Now if $P_f^n \cap J_i = \varnothing$ then $P_g^n \cap J_i = \varnothing$ or else $P_g^{n\cdot k} \cap J_i \neq \varnothing$. Thus $g \in B(f,\varepsilon') \cap Q_k^\varepsilon$ and it is easy to see that if $\eta = v/10$ then $B(g,\eta) \subset B(f,\varepsilon') \cap Q_k^\varepsilon$. It follows that Q_k^ε contains

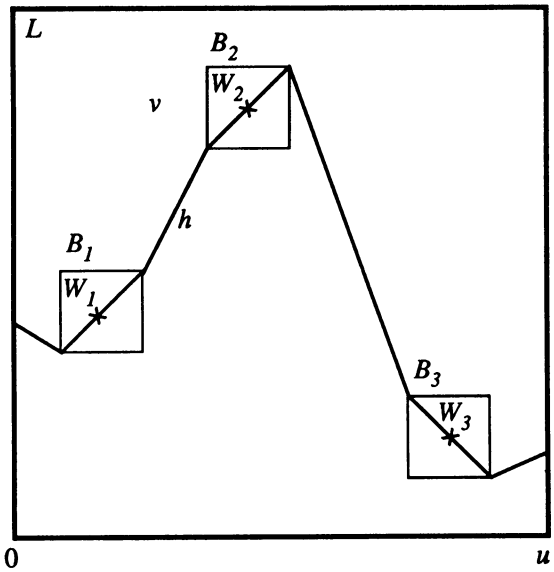

FIGURE 2

an open dense set, and since $\bigcap_{\varepsilon,k} Q_k^\varepsilon \subset A$ we have that A is a residual subset of C. This completes the proof of the theorem.

We show that, for a typical continuous f, P_f^k is a dense relatively open subset of $\mathrm{Fix}(f^k)$ from which it will follow that, P_f^k is a residual subset of $\mathrm{Fix}(f^k)$ and P_f^k is uncountable.

Corollary 1. *For a typical continuous f, P_f^k is a dense relatively open subset of* $\mathrm{Fix}(f^k)$.

Proof. If $x \in \mathrm{Fix}(f^k)$ then there is $q|_k$ such that $x \in P_f^q$. It follows from Theorem 1 that any neighborhood of x contains periodic points with period $q \cdot k/q$. As we have seen in the introduction, P_f^k is relatively open in $\mathrm{Fix}(f^k)$.

Corollary 2. *For a typical continuous f, P_f^k is not closed for every $k > 1$.*

Proof. It follows from Theorem 1 that for every $x \in \mathrm{Fix}(f)$, $x \in \mathrm{cl}(P_f^k)$ and $x \notin P_f^k$.

Corollary 3. *For a typical continuous f, P_f^k is dense in itself.*

Proof. Since, for a typical f, $\mathrm{Fix}(f^k)$ is dense in itself [1], the statement follows from Corollary 1.

Remark. For a typical f there is $x \in \mathrm{Fix}(f^n)$ such that in a suitable neighborhood of x, the periods of the periodic points are all multiples of n. Moreover, this is true for the points of a residual subset of $\mathrm{Fix}(f^n)$. Indeed, let $f \in C$ and $I_0 \subset [0,1]$ be an open interval and let $I_k = f^k(I_0)$. Define $P_k^n(f) = \{I_0\colon$ for every $0 \le 1 \ne j \le k-1$, $I_i \cap I_j = \varnothing$ and $I_n \subset I_0$ and $\min\{|I_0|, \dots, |I_n|\} < 1/k$. (See [1, Proposition 1].)

In [1] the authors proved that for a typical continuous function f, $\mathrm{Fix}(f^n) \cap \bigcap_{k=1}^{\infty} G_k^n$ is a residual subset of $\mathrm{Fix}(f^n)$. Put $B_f^n = \{x \in P_f^n\colon$ there is V neighborhood of x such that for any periodic point $y \in V$ there is k such that period of y equal to $n \cdot k\}$. It is easy to see $P_f^n \cap \bigcap_{k=1}^{\infty} G_k^n \subset B_f^n$ and therefore B_f^n is a residual subset of $\mathrm{Fix}(f^n)$.

In [1] it is proved that for a typical continuous f the set $\mathrm{Fix}(f^k)$ is nowhere dense and perfect. In Theorem 2 we shall prove that for every porosity premeasure (i.e., a continuous map $\Phi\colon (0,1] \to (0,1]$), $\mathrm{Fix}(f^k)$ is bilaterally strongly Φ-porous for a typical continuous f. This means that for every $x \in \mathrm{Fix}(f^k)$ there are sequences of intervals $\{I_n\}_{n=1}^{\infty}$ and $\{J_n\}_{n=1}^{\infty}$ such that $I_n \subset (x - 1/n, x) \setminus \mathrm{Fix}(f^k)$, $J_n \subset (x, x + 1/n) \setminus \mathrm{Fix}(f^k)$ and

$$\lim_{n\to\infty} \frac{\mathrm{dist}(x, I_n)}{\Phi(|I_n|)} = \lim_{n\to\infty} \frac{\mathrm{dist}(x, J_n)}{\Phi(|J_n|)} = 0.$$

We shall apply the method used by P. Humke and M. Laczkovich in [2] to this problem.

The pair of sequences (α, β) is said to be proper if $\alpha = \{\alpha_n\}_{n=1}^{\infty}$, $\beta = \{\beta_n\}_{n=1}^{\infty}$ and $\beta_n \to 0$ and $0 < \alpha_n < \beta_n$. If (α, β) is a proper pair of sequences then the sequence $x = \{x_n\}_{n=1}^{\infty}$ is called an (α, β)-sequence if $x_n \to x_0$ and $\alpha_n \le x_n - x_0 \le \beta_n$ for each $n \in \mathbf{N}$. Let the natural number $l > 1$ and the proper pair (α, β) be fixed. (See [2, p. 245].) Put

$B_N = \{f \in C\colon$ there is $\{x_i\}_{i=1}^{\infty}(\alpha, \beta)$-sequence such that $f^l(x_i) = x_i$ if $i \ge N\}$.

Lemma. *B_N is nowhere dense.*

Proof. First we shall prove that B_N is closed. Let $\{f_k\}_{k=1}^{\infty} \subset B_N$ and $f_k \to f$ in C. Thus, for every fixed k there is a sequence $x^k = \{x_n^k\}_{n=1}^{\infty}$ converging to x_0^k such that x^k is an (α, β)-sequence and $f_k^l(x_i^k) = x_i^k$ if $i \ge N$. Using a diagonal procedure we obtain a subsequence $\{k_i\}_{i=1}^{\infty}$ such that for every n there is x_n^* such that $\{x_n^{k_i}\}_{i=1}^{\infty} \to x_n^*$ and $x^* = \{x_n^*\}_{n=1}^{\infty}$ is a (α, β)-sequence. As $\{f_k\}_{k=1}^{\infty} \to f$ in C and since for $n > N$ $f_{k_i}^l(x_n^{k_i}) = x_n^{k_i}$ it follows that $f^l(x_n^*) = x_n^*$ if $n > N$.

If p is a polynomial then $\{x\colon p^l(x) = x\}$ is finite and hence $p \notin B_N$. Since the set of polynomials is dense in C, the complement of B_N is dense and, as B_N is closed, B_N is nowhere dense.

Let $\Phi\colon (0,1] \to (0,1]$ be continuous arbitrary fixed.

Theorem 2. *For a typical continuous* f, $\mathrm{Fix}(f^l)$ *is bilaterally strongly* Φ*-porous set.*

Proof. Let (α, β) be such that $\alpha_n = \beta_{n+1}$, $\beta_n < 1/n$ and $\Phi(\beta_n - \beta_{n+1}) \geq n\beta_{n+1}$. Let $M = \{x: f^l(x) = x\}$. If $f \in \bigcup_n B_n$ then for each $x \in M$ there is a sequence $\{n_i\} \to \infty$ such that $M \cap [x + \alpha_{n_i}, x + \beta_{n_i}] = \varnothing$. Let $J_i = [x + \alpha_{n_i}, x + \beta_{n_i}]$. Thus $J_i \subset (x, x + 1/i)\backslash M$ and

$$\mathrm{dist}(x, J_i) = \alpha_{b_i} = \beta_{n_i+1} < \frac{\Phi(\beta_{n_i} - \beta_{n_i+1})}{n_i} = \frac{\Phi(|J_i|)}{n_i}$$

and hence

$$\lim_{n\to\infty} \frac{\mathrm{dist}(x, J_i)}{\Phi(|J_i|)} = 0.$$

Using a similar procedure we can find a sequence $\{I_i\}_{i=1}^{\infty}$ such that

$$I_i \subset (x - 1/i, x)\backslash M \quad \text{and} \quad \lim_{i\to\infty} \frac{\mathrm{dist}(x, I_i)}{\Phi(|I_i|)} = 0.$$

This completes the proof of the theorem.

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