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KAPLANSKY'S PROBLEM ON VALUATION RINGS

LASZLO FUCHS AND SAHARON SHELAH

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Dedicated to Irving Kaplansky on his seventieth birthday

ABSTRACT. The following theorem is proved in ZFC: there exist valuation rings which are not surjective homomorphic images of valuation domains. The proof relies on the existence of nonstandard divisible uniserial modules in ZFC.

Let R be a valuation ring (in another terminology: a chained ring), i.e. a commutative ring with 1 in which the ideals form a chain under inclusion. A valuation domain is a domain that is a valuation ring at the same time.

Kaplansky raised the question as to whether or not every valuation ring R can be obtained as a (surjective) homomorphic image of a valuation domain S. Under additional conditions on R, the answer is affirmative; see Ohm and Vicknair [3] and the literature cited there. On the other hand, Fuchs and Salce [2, p. 151] have given an explicit example for a valuation ring which can not be obtained in the indicated way. Their proof was based on the existence of nonstandard divisible uniserial modules by using Jensen's Diamond Principle which holds in the constructible universe but not in ZFC alone (Zermelo-Fraenkel set theory with the Axiom of Choice). Franzen and Göbel [1] pointed out that the slightly weaker hypothesis $2^{\aleph_0} < 2^{\aleph_1}$ suffices. (The existence of nonstandard uniserials was first established by Shelah [5] by using forcing argument; later he replaced it by an absoluteness result of stationary logic.)

Our aim here is to prove in ZFC the existence of nonstandard divisible uniserial modules and the existence of valuation rings which are not obtainable as

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homomorphic images of valuation domains. We also improve on the algebraic part of the argument.

1. Preliminaries on valuation rings

In a valuation ring R, the nilradical N coincides with the set of nilpotent elements of R. This N is the minimal prime of R.

A relevant information about the powers of primes is the content of the next lemma. Here *uniserial* means that the submodules form a chain.

Lemma 1. *Let P be a prime ideal of the valuation ring R such that $P^2 \neq P$. If Q^* denotes the field of quotients of $R^* = R/P$, then*

(1) *for each $n \geq 2$, P^{n-1}/P^n is a divisible uniserial R^*-module;*

(2) *if $P^n \neq 0$, then $P^{n-1}/P^n \cong_{R^*} Q^*$.*

Proof. Evidently, P^{n-1}/P^n is an R^*-module. To verify that it is uniserial, pick $x, y \in P^{n-1} \backslash P^n$. Then $y = rx$ for some $r \in R$ (or vice versa). $y \notin P^n$ guarantees that $r \notin P$, hence $y + P^n \in R^*(x + P^n)$.

Observe that for a prime ideal P of a valuation ring R, $rP = P$ holds for all $r \in R \backslash P$ (see e.g. [2, p. 15]). Hence $rP^n = P^n$ for every $n \geq 1$ and $r \in R \backslash P$, and the R^*-divisibility of P^{n-1}/P^n follows.

Turning to the proof of (2), suppose $rx = y \in P^n$ for some $r \in R \backslash P$ and $x \in P^{n-1} \backslash P^n$. In view of $rP^n = P^n$, we can write $y = rz$ $(z \in P^n)$ where $z = tx$ for some $t \in R$. This t cannot be a unit, so $1 - t$ is a unit, and $r(1-t)x = 0$ implies $rx = 0$. We infer that r annihilates Rx, and hence its submodule $P^n = rP^n$. Consequently, $P^n = 0$, and all P^{n-1}/P^n with $P^n \neq 0$ are torsion free as R^*-modules. The proof can be completed by observing that a torsion-free divisible uniserial R^*-module is necessarily isomorphic to Q^*.

Applying the preceding lemma to $P = N$, we conclude that in the descending chain $N > N^2 > \cdots > N^{n-1} > N^n = 0$, all factors are isomorphic to Q^* with the possible exception of the last one: N^{n-1}. Information on N^{n-1} is given by

Lemma 2. *If the valuation ring R is a homomorphic image of a valuation domain, and if its nilradical N satisfies $N^{n-1} \neq 0 = N^n$, then N^{n-1} is an epic image of Q^*.*

Proof. Let $\phi: S \to R$ be a surjective homomorphism, S a valuation domain. $\phi^{-1}N = P$ is a prime ideal of S such that $\phi P^i = N^i$ for every i. Clearly, ϕ induces an R^*-epimorphism $P^{n-1}/P^n \to N^{n-1}/N^n$. Since S is a domain, $P^n \neq 0$, so by Lemma 1, the first module is isomorphic to Q^*.

Let R^* be a valuation domain and U^* a divisible uniserial R^*-module. Following [2], we call U^* *standard* if it is an epic image of Q^*; otherwise U^* is *nonstandard*. Form the R^*-module

$$R = R^* \oplus U^* \tag{1}$$

and define multiplication in R via

$$(r, u) \cdot (s, v) = (rs, su + rv) \qquad (r, s \in R^*; u, v \in U^*).$$

We then obtain a valuation ring R with nilradical $N = U^*$ where $N^2 = 0$.

The "if" part of the next lemma is based on an observation by J. Ohm.

Lemma 3. *The valuation ring* (1) *is a homomorphic image of a valuation domain if and only if* U^* *is a standard uniserial divisible* R^**-module.*

Proof. Necessity is obvious from Lemma 2. Conversely, let $U^* \cong Q^*/K^*$ for an ideal K^* of R^*. Consider the formal power series ring S in $Q^*[[t]]$ consisting of all formal power series of the form

$$a_0 + a_1 t + \cdots + a_m t^m + \cdots$$

with $a_0 \in R^*$, $a_m \in Q^*$ $(m \geq 1)$. It is readily seen that either $R \cong S/St^2$ or $R \cong S/K^*St$ according as $K^* = 0$ or $\neq 0$.

2. Divisible uniserial modules

To solve Kaplansky's problem, we proceed to verify the existence of valuation domains R which admit nonstandard divisible uniserial modules. We accomplish this goal by relying on results in Shelah [4]. We emphasize that we are working in ZFC.

Consider the class **K** consisting of (multisorted) models $N = (L^N, Q^N, U^N, T^N, f^N, g^N)$ where

(i) L^N is a linearly ordered set without largest element,

(ii) Q^N is the field of quotients of a valuation domain R^N with the field operations and a predicate for R^N

$$f^N : L^N \to R^N \backslash 0$$

satisfying

(α) $s < t$ in L^N implies that $f^N(s)$ divides $f^N(t)$ in R^N;

(β) for every $a \in R^N \backslash 0$ there is an $s \in L^N$ such that a divides $f^N(s)$.

[Note that these conditions assure that $\{f^N(s)R^N | s \in L^N\}$ is a chain of ideals with 0 intersection.]

(iii) U^N is a divisible uniserial torsion R^N-module with a function

$$g^N : L^N \to U^N$$

satisfying

(γ) $s < t$ in L^N implies $g^N(s)R^N \leq g^N(t)R^N$ in U^N;

(δ) for each $s \in L^N$, we have

$$\operatorname{Ann} g^N(s) = \{r \in R^N | rg^N(s) = 0\} = f^N(s)R^N.$$

[Observe that the union of the submodules $g^N(s)R^N (s \in L^N)$ is divisible, hence equals U^N, and the elements of U^N have principal ideal annihilators.]

(iv) T^N is a tree (i.e. a partially ordered set such that $\{y|y \leq x\}$ is linearly ordered for every x) with levels T_t^N indexed by the elements t of L^N, more explicitly, this means that $x_1 < x_2 (x_i \in T_{t_i}^N)$ implies $t_1 < t_2$ in L^N, and $t_1 < t_2$ in L^N, $x_2 \in T_{t_2}^N$ imply there is a unique $x_1 \in T_{t_1}^N$ with $x_1 < x_2$, and the relation $\{(t, x)|t \in L^N, \ x \in T_t^N\}$ is one of the relations of T^N. T_t^N is defined to be the set of all isomorphisms

$$\phi_t \colon f^N(t)^{-1} R^N / R^N \to g^N(t) R^N$$

between the indicated submodules of Q^N/R^N and U^N, where the partial order is defined by setting $\phi_s \leq \phi_t$ exactly if $s \leq t$ and ϕ_s is a restriction of ϕ_t.

The following lemma is a straightforward consequence of the definition.

Lemma 4. $\mathbf{K}$ *is the class of models of a first order theory* Γ. $\square$

For $N \in \mathbf{K}$, a subset B^N of the tree T^N is called a *full branch* of T^N if it is a linearly ordered subset of T^N which contains exactly one element at each level T_t^N, i.e. $|B^N \cap T_t^N| = 1$ for each $t \in L^N$.

Lemma 5. *For* $N \in \mathbf{K}$, *the uniserial* R^N*-module* U^N *is isomorphic to* Q/R (*and hence standard*) *if and only if* T^N *contains a full branch.*

Proof. If $\psi \colon Q/R \to U^N$ is an isomorphism, then the restrictions ψ_t of ψ to $f^N(t)^{-1} R^N / R^N$ form a full branch of T^N. On the other hand, if B^N is a full branch of T^N, then $\psi = \bigcup\{\psi_t | t \in B^N\}$ defines an isomorphism $Q/R \to U^N$.

Our final preparatory lemma is concerned with certain models of ZFC.

Lemma 6. *Every model* V *of ZFC has a generic extension in which, for some* $N \in \mathbf{K}$, U^N *is a nonstandard uniserial* R^N*-module.*

Proof. As is shown in Fuchs-Salce [2, p. 149], ZFC $+\diamondsuit_{\aleph_1}$ implies that a model $N \in \mathbf{K}$ with U^N nonstandard uniserial does exist. If κ is a regular uncountable cardinal, then forcing V with Lévy $(\aleph_1, 2^{\aleph_0}), \diamondsuit_{\aleph_1}$ will be satisfied—as is well known. Hence the claim is immediate.

Another proof for Lemma 6 can be given by using the method applied in Shelah [5].

We have come to the main existence lemma.

Lemma 7. *For every regular cardinal* λ, *there is a model* $N \in \mathbf{K}$ *such that both* R^N *and* U^N *have cardinalities* λ^+ *and* U^N *is a nonstandard divisible uniserial* R^N*-module.*

Proof. For any first-order formula $\phi(x, \overline{y})$ in function symbols and predicates from the vocabulary τ of $\mathbf{K}$ only, there is a first-order formula $\psi_\phi(\overline{y})$ such that for $N \in \mathbf{K}$ and a finite sequence $\overline{a}$ from N of the length of $\overline{y}$

(*) $\quad N \models \psi_\phi[\overline{a}]$ if and only if $\{b \in N | N \models \phi(b, a)\}$ is a full branch of T^N.

Let $\Gamma^+ = \Gamma \cup \{\neg(\exists \overline{y})\psi_\phi(\overline{y}) | \phi(x, \overline{y})$ is a first-order formula in the vocabulary $\tau\}$. Now Γ^+ is a first-order theory. After the forcing, it has a model (see Lemma 6) and hence is consistent, i.e. by Göbel's completeness theorem, there is no finite proof of contradiction from Γ^+. But any such proof is a finite sequence of formulas, hence it is from the universe before the forcing. It follows that Γ^+ is consistent in V (this is a well-known absoluteness theorem). Now by Theorem 12 in [4, p. 81], Γ^+ has a model N of cardinality λ^+ in which T^N has no full branches except for those definable in N by some first-order formula with parameters from V. But by the definition of Γ^+, there are no such formulas. Now the proof of Theorem 12 of [4] gives that every subset of N definable by a first-order formula with parameters is of size λ^+; alternatively use $\Gamma^* = \Gamma^+ \cup \{\psi^1, \psi^2\}$ where ψ^j says that F_j is a one-to-one function from N to R^N and U^N, respectively. By Lemma 5, U^N is nonstandard. □

Another proof can be given by using the following argument. Denote by Γ the first-order theory in which **K** is a class of models; such a Γ exists in view of Lemma 4. Lemma 6 guarantees that in some generic extension of V, Γ has a model N for which the tree T^N fails to have a full branch. This means that if $\mathscr{L}$ is the first-order logic expanded by a quantifier on full branches of trees (see Application D in [4, p. 74]; of course $L(Q_{Br})$ satisfies the completeness theorem by Theorem 12 of [3, p. 81]), then there is a $\psi \in \mathscr{L}$ saying that T^N has no full branch. Consequently, $\Gamma \cup \{\psi\}$ has a model in some generic extension. Therefore, it is consistent in V. We refer to [4] to conclude that $\Gamma \cup \{\psi\}$ has in V a model N of cardinality λ^+ (in which all nonfinite definable sets have cardinality λ^+). Again by Lemma 5, U^N is nonstandard.

3. The existence theorem

We can now put the pieces together to prove our main result, answering Kaplansky's problem.

Theorem (ZFC). *There exist valuation rings (the squares of whose nilradicals vanish) which are not homomorphic images of valuation domains.*

Proof. By Lemma 7, there exist valuation domains R^* which admit nonstandard divisible uniserial R^*-modules U^*. An appeal to Lemma 3 completes the proof.

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DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LOUISIANA 70118

DEPARTMENT OF MATHEMATICS, THE HEBREW UNIVERSITY, JERUSALEM, ISRAEL