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RULED FANO 4-FOLDS OF INDEX 2

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ABSTRACT. This article contains a classification of Fano 4-folds of index 2 which can be represented as P^1-bundles.

Let X be a smooth projective variety of dimension $n \geq 1$ over the field of complex numbers. We call X a Fano n-fold if its anticanonical divisor $-K_X$ is ample. The index $r(X)$ of a Fano n-fold X is defined as

$$r(X) = \max\{k\colon -K_X = kH \text{ for } H \text{ ample divisor on } X\}.$$

We will say that X is ruled if $X = P(\mathscr{E})$ for some rank-2 vector bundle $\mathscr{E}$ on a projective manifold M. The bundle $\mathscr{E}$ is called Fano if $X = P(\mathscr{E})$ is a Fano n-fold. Together with Michal Szurek we worked on some Fano bundles of rank-2. In [SzW1] we classified all ruled Fano 3-folds and in [SzW2] we described all rank-2 Fano bundles on P^3. The purpose of this paper is to classify all ruled Fano 4-folds of index 2.

(0.1) **Theorem.** *Assume that X is a ruled Fano 4-fold of index 2. Then one of the following holds*:

(i) $X = P^1 \times M$, *where M is a Fano 3-fold of index 2 or P^3*;

(ii) *either $X = P(\mathscr{O}_{P^3}(1) \oplus \mathscr{O}_{P^3}(-1))$ or $X = P(\mathscr{O}_{Q^3} \oplus \mathscr{O}_{Q^3}(-1))$*;

(iii) *X has two P^1-bundle stuctures and can be realized either as $P(NCB)$, where NCB is the null-correlation bundle on P^3, that is, a stable rank-2 bundle with $c_1 = 0$, $c_2 = 1$, or $P(\mathscr{E})$, where $\mathscr{E}$ is a stable rank-2 bundle on Q^3 with $c_1\mathscr{E} = -1$, $c_2\mathscr{E} = 1$.*

The techniques used in this paper are a mixture of those from [SzW1 and SzW2], and others coming from contractions of extremal rays on Fano manifolds. In §1 we outline properties of contractions of extremal rays and prove a useful lemma on contractions of extremal rays of length $\geq \max(2, n-2)$ (Lemma (1.1)). In §§2 and 3 we give the proof of Theorem (0.1).

Remark. After the first version of this paper was completed, I learned that the classification of Fano 4-folds of index 2 was the subject of a preprint of S. Mukai

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[*On Fano manifolds of coindex* 3]. His list, containing broader class of Fano manifolds, was obtained by different methods under the assumption that the linear system $|-\frac{1}{2}\cdot K_X|$ contains a smooth divisor.

1. Contractions of extremal rays.

First, let us recall briefly the terminology and results about contractions of extremal rays. We refer the reader to [Mo and An] for details.

Assume that X is Fano n-fold, $n \geq 2$. Let $N_1(X)$ denote the space

$$(\{1\text{-cycle on } X\text{, module numerical equivalence}\}) \otimes R.$$

Inside $N_1(X)$ we have a convex cone $NE(X)$ spanned on the classes of all effective 1-cycles. The Mori Cone Theorem [Mo, Theorem (1.2)] yields that $NE(X)$ is a rational polyhedron. The edges of $NE(X)$ are called extremal rays. Every extremal ray R is spanned by the class of a rational curve, and its length, defined as $l(R) = \min\{-K_X \cdot C\colon\ C$ rational curve whose numerical class is in $R\}$ is at most $n+1$. A rational curve C in R is called an extremal rational curve if $-K_X \cdot C = l(R)$.

The Kawamata-Shokurow Contraction Theorem (cf. Corollary 1.3 of [An]) yields that for any extremal ray R there exist a normal projective variety Y and an epimorphism

$$\mathrm{contr}_R\colon X \to Y$$

such that all fibers of contr_R are connected, and a curve C on X is contracted by contr_R to a point if and only if its class is in R.

A numerically effective (nef) divisor D_R is called a good supporting divisor for R if, for $Z \in NE(X)$, $D_R \cdot Z = 0$ if and only if $Z \in R$. We call an extremal ray R numerically effective if $D \cdot R \geq 0$ (i.e. $D \cdot Z \geq 0$ for any $Z \in R$) for any effective divisor D on X.

The following conditions are equivalent:

(i) R is nef;
(ii) $D_R^n = 0$, for a good supporting divisor D_R for R;
(iii) $\dim(\mathrm{contr}_R(X)) < n$.

Now let us prove a result on a contraction of an extremal ray of length $\geq \max(2, n-2)$. Note that if X is a Fano 4-fold of index 2 then every extremal ray of X satisfies this inequality.

(1.1) **Lemma.** *Let R be an extremal ray of X whose length is $\geq \max(n-2, 2)$. If R is not nef then there exists a unique prime divisor $E \subset X$ such that $E \cdot R < 0$. Moreover, we have the following inequality:*

$$\dim(\mathrm{contr}_R(E)) \leq n-3.$$

Proof. From the Ionescu estimate [Io, Theorem 0-4] it follows that the dimension of the locus of curves in R is at least $n-1$. Thus, if R is not nef, the dimension of the exceptional set of contr_R is $n-1$, and we are in the situation discussed by Ando in §2 of [An]. Therefore $E =$ (locus of curves in R) is a

unique prime divisor such that $E \cdot R < 0$. If $\dim(\mathrm{contr}_R(E)) = n-2$, then a general fiber of $\mathrm{contr}_{R|E}$ is isomorphic to P^1 [An, Theorem 2.1], and we see that its intersection with $-K_X$ is equal to 1 (cf. proof of Theorem 2.3 in [An]), which contradicts the assumption that the length of R is at least 2.

(1.2) **Corollary.** *Let D_R be a good supporting divisor for an extremal ray R of length $\geq \max(2, n-2)$. Then either R is nef and $D_R^n = 0$, or there exists a prime divisor E such that the 1-cycle $D_R^{n-2}E$ is numerically trivial.*

Remark. We do not have to assume that X is Fano to get (1.1) and (1.2). Actually, it is enough to assume that K_X is not nef and extremal rays are defined as in [Mo].

From now on, let X be a Fano 4-fold of index 2. The following lemmas are derived from (1.1).

(1.3) **Lemma.** *Let $\mathrm{contr}_R: X \to Y$ be a contraction of an extremal ray R of X. If every fiber of contr_R is of dimension ≤ 1, then X is ruled.*

Proof. From (1.1) it follows that R is nef. Therefore we are in the situation of Theorem 3.1(ii) from [An]. Therefore $\mathrm{contr}_R: X \to Y$ is a conic bundle. But on X there exists an ample divisor H (such that $-K_X = 2H$), whose intersection with any fiber of contr_R equals to 1, hence $\mathrm{contr}_R: X \to Y$ is a P^1-bundle.

(1.4) **Lemma.** *Assume that $b_2(X) \geq 2$. If there exists an extremal ray R of X whose contraction has a 3-dimensional fiber then X is ruled.*

Proof. First, let us consider the case when R is nef. Let F be a prime divisor on X contracted by contr_R to a point. The divisor F does not meet other fibers of contr_R containing curves from R, hence $F \cdot R = 0$. There exists an extremal ray $R' \neq R$ such that $F \cdot R' > 0$. We claim that all fibers of $\mathrm{contr}_{R'}$ are at most 1-dimensional, which by the previous lemma proves X being ruled. Indeed, a fiber of $\mathrm{contr}_{R'}$ of dimension ≥ 2 would meet F at at least 1-dimensional set, therefore we would have a curve on X contracted by both contr_R and $\mathrm{contr}_{R'}$, which is impossible.

Now assume that R is not nef. Let E be the divisor from Lemma (1.1). Now we can choose an extremal ray R' such that $E \cdot R' > 0$, and by the same argument as above, we see that $\mathrm{contr}_{R'}$ gives a ruling of X.

The following corollary (which will not be used in the subsequent sections) can be easily derived from the Theorem (0.1) and the proof of (1.4):

(1.5) **Corollary.** *If there exists a morphism from X onto a curve then $X = P^1 \times M$, where M is a Fano 3-fold of index 2 or P^3.*

2. Ruled Fano 4-folds of index 2, $b_2 \geq 3$.

From now on assume that X is a Fano 4-fold of index 2 and $X = P(\mathscr{E})$ for some rank-2 vector bundle on a smooth 3-fold M. Let p denote the

projection morphism from $X = P(\mathscr{E})$ onto M. On X we have a relative ample line bundle $\xi_{\mathscr{E}}$ such that its restriction to every fiber of p is isomorphic to $\mathscr{O}_{P^1}(1)$ and $p_*\xi_{\mathscr{E}} = \mathscr{E}$. Replacing $\mathscr{E}$ by its twist with a line bundle $\mathscr{L}$ on M does not affect the projectivization and $\xi_{\mathscr{E}\otimes\mathscr{L}} = \xi_{\mathscr{E}} \otimes p^*(\mathscr{L})$. The 3-fold M is Fano (cf. [SzW1, (1.7)]). Let D be an ample divisor on M such that

$$-K_M = r(M)D.$$

The anticanonical divisor of X reads as

$$-K_X = 2\xi_{\mathscr{E}} - c_1\mathscr{E} + r(M)D \tag{2.1}$$

where c_1E and D denote, by abuse of notation, $p^*(c_1\mathscr{E})$ and $p^*(D)$, respectively. Let H be an ample divisor on X such that $-K_X = 2H$.

We will use the following lemma [SzW1, (1.5)]:

(2.2) **Lemma.** *Let $C \subset M$ be a rational curve with normalization $v\colon P^1 \to C \subset X$. If $v^*\mathscr{E} = \mathscr{O}_{P^1}(a_1) \oplus \mathscr{O}_{P^1}(a_2)$ then*

$$|a_1 - a_2| < -K_M \cdot C.$$

We use this fact in proving

(2.3) **Lemma.** *If $r(M) = 1$ then $M = P^1 \times P^2$ and X is either*

$$P^1 \times P(\mathscr{O}_{P^2} \oplus \mathscr{O}_{P^2}(-1))$$

or

$$P^1 \times P(TP^2(-2))).$$

Proof. First let us assume that $r(M) = 1$ and M is not $P^1 \times P^2$. It is known (cf. [Sh and MM]) that there exists a rational curve $C \subset M$ such that $-K_M \cdot C = D \cdot C = 1$. Then, by (2.2), it follows that $c_1\mathscr{E} \cdot C$ must be an even number. This is impossible because $-K_X$ satisfying (2.1), has to be divisible by 2.

Now let $M = P^1 \times P^2$. Since twisting of $\mathscr{E}$ by a line bundle on M does not affect its projectivization, we can assume that $c_1\mathscr{E} \cdot (P^1 \times \{x\}) = 0$ or 1, for any $x \in P^2$. Again, using (2.2), we see that $\mathscr{E}|_{(P^1\times\{x\})}$ is trivial for any $x \in P^2$ because $c_1\mathscr{E}|_{(P^1\times\{x\})}$ is even. Thus $\mathscr{E}$ is a pull-back to $P^1 \times P^2$ of a bundle $\mathscr{E}'$ from P^2, hence $X = P^1 \times P(\mathscr{E}')$. From [SzW1] it follows that $\mathscr{E}'$ can be chosen to be either $\mathscr{O}_{P^2} \oplus \mathscr{O}_{P^2}(-1)$ or $TP^2(-2)$.

From now on we assume $r(M) \geq 2$, therefore M is one of the following:

(i) P^3;
(ii) Q^3;
(iii) $P^1 \times P^1 \times P^1$;
(iv) $V =$ blow-up of P^3 at a point $= P(\mathscr{O}_{P^2} \oplus \mathscr{O}_{P^2}(-1))$;
(v) $W =$ divisor on $P^2 \times P^2$ of type $(1, 1) = P(TP^2(-2))$;
(vi) one of the Fano 3-folds V_d, $d = 1, \dots, 5$ listed in [Is1]; $b_2(V_d) = 1$.

Note that if M is any of the above manifolds and C_0 an extremal rational curve on M, then there exists a divisor D_0 such that $D_0 \cdot C_0 = 1$ and $D_0 \cdot C = 0$ for any extremal rational curve C not equivalent to C_0. Therefore, since twisting of $\mathscr{E}$ by a line bundle on M does not affect projectivization, we can assume that $\mathscr{E}$ is normalized, i.e. $c_1\mathscr{E} \cdot C = 0$ or -1 for any extremal rational curve C on M. Moreover, since $-K_X$ is divisible by 2, we have

$$c_1\mathscr{E} = -D \text{ if } M = Q^3, \quad \text{and} \quad c_1\mathscr{E} = 0 \text{ otherwise.} \tag{2.4}$$

We conclude this section with

(2.5) **Lemma.** *If* $b_2(M) \geq 2$ *then* $X = P^1 \times M$.

PROOF. We see that M is one of the following:

$$P^1 \times P^1 \times P^1, \qquad V = P(\mathscr{O}_{P^2} \oplus \mathscr{O}_{P^2}(-1)), \qquad W = P(TP^2(-2)).$$

In either of these cases we have a morphism $\Psi\colon M \to S$, where S is either P^2 or $P^1 \times P^1$, that makes M a P^1-bundle. By (2.4) and (2.2) it follows that $\mathscr{E}$ restricted to any fiber of Ψ is trivial. Thus $\mathscr{E}' = \Psi_*\mathscr{E}$ is a rank-2 vector bundle on S and $\mathscr{E} = \Psi^*\mathscr{E}'$. We claim that $\mathscr{E}'$ is a trivial bundle. Indeed, let L be a line on S. Over L we have a section C of Ψ that is an extremal rational curve on M. Again, by (2.4) and (2.2) it follows that $\mathscr{E}_{|C}$ is trivial. Now we have

$$\mathscr{E}'_{|L} \simeq \Psi^*\mathscr{E}'_{|C} \simeq \mathscr{E}_{|C} \simeq \mathscr{O}_C \oplus \mathscr{O}_C$$

hence $\mathscr{E}'_{|L}$ is trivial, and by a well-known property of vector bundles on P^2 and $P^1 \times P^1$ it follows that $\mathscr{E}'$ is trivial.

3. RULED FANO 4-FOLD OF INDEX 2, $b_2 = 2$.

Let $X, M, \mathscr{E}, H, D$ be as in the previous section. We moreover assume that $r(M) \geq 2$, $b_2(M) = 1$ and $\mathscr{E}$ is normalized, i.e. satisfies (2.4).

Let c be an integer such that

$$c_2(\mathscr{E}) = c \cdot (\text{class of a line on } M).$$

Knowing the Riemman-Roch theorem on M (see for example [Is2]) we compute the Euler characteristic of $\mathscr{E}$:

(i) if $M = P^3$ then $\chi(\mathscr{E}) = 2 - 2c$;
(ii) if $M = Q^3$ then $\chi(\mathscr{E}) = 1 - c$;
(iii) if $r(M) = 2$ then $\chi(\mathscr{E}) = 2 - c$.

The Leray-Hirsch formula yields the following equalities on X (cf. [SzW1 and SzW2]):

$$D^4 = D^2 \cdot \xi_{\mathscr{E}}^2 = \xi_{\mathscr{E}}^4 = 0, \quad D^3 \cdot \xi_{\mathscr{E}} = d(M), \quad D \cdot \xi_{\mathscr{E}}^3 = -c \tag{3.1}$$

if $c_1\mathscr{E} = 0$ and $d(M)$ denotes the self-intersection of D on M;

$$\begin{gathered} D^4 = 0, \quad D^3 \cdot \xi_{\mathscr{E}} = 2, \quad D^2 \cdot \xi_{\mathscr{E}}^2 = -2, \quad D \cdot \xi_{\mathscr{E}}^3 = 2 - c, \\ \xi_{\mathscr{E}}^4 = 2(c-1) \quad \text{if } M = Q^3 \text{ and } c_1\mathscr{E} = -D. \end{gathered} \tag{3.2}$$

Now, since $(-K_X)^4 > 0$, we have the following bounds on c:

$$c < 4 \quad \text{if } M = P^3; \tag{3.3}$$

$$c < 5 \quad \text{if } M = Q^3; \tag{3.4}$$

$$c < d(M) \quad \text{if } r(M) = 2. \tag{3.5}$$

On the other hand we have

(3.6) **Lemma.** *If $c \leq 0$ then $\mathscr{E}$ is decomposable and either $X = P^1 \times M$ or $X = P(\mathscr{O}_{P^3}(-1) \oplus \mathscr{O}_{P^3}(1))$ or $X = P(\mathscr{O}_{Q^3} \oplus \mathscr{O}_{Q^3}(-1))$.*

Proof. We see that if $c \leq 0$, then $\chi(\mathscr{E}) > 0$. On the other hand the bundle $\mathscr{E} \otimes \mathscr{O}(-K_M)$ is ample by (2.1), hence by Le Portier vanishing (cf. [SS, (5.17)]), $H^i(M, \mathscr{E}) = 0$ for $i \geq 2$. Thus $\mathscr{E}$ has a nontrivial section. If this section does not vanish anywhere, then $c = 0$ and we see that $\mathscr{E} = \mathscr{O}_M \oplus \mathscr{O}_M$ if $c_1\mathscr{E} = 0$, or $\mathscr{E} = \mathscr{O}_{Q^3} \oplus \mathscr{O}_{Q^3}(-1)$ if $c_1\mathscr{E} = -D$.

Assume now that Z is a nonempty zero set of a nontrivial section of $\mathscr{E}$. If C is an extremal rational curve that meets Z, and $\mathscr{E}_{|C} = \mathscr{O}_C(a_1) \oplus \mathscr{O}_C(a_2)$, then $\max(a_1, a_2) > 0$. Now from Lemma (2.2) it follows that $M = P^3$ and from the classification from [SzW2] we see that $\mathscr{E} = \mathscr{O}_{P^3}(-1) \oplus \mathscr{O}_{P^3}(1)$.

Now we will apply Corollary (1.2) to find possible positive values of c. Let D_R be a good supporting divisor for an extremal ray R that is not contracted by $p\colon X \to M$. The extremal ray R is either nef or there exists a prime divisor E as in Lemma (1.1). In any case, both, D_R and E, are positive multiples of $(\xi_{\mathscr{E}} + uD)$ and $(\xi_{\mathscr{E}} + vD)$, respectively, where u and v are rational numbers.

If R is nef then:

(3.7) There exists a rational solution u_0 of the equation

$$(\xi_{\mathscr{E}} + uD)^4 = 0$$

such that for any $u_1 > u_0$

$$(\xi_{\mathscr{E}} + u_1 D)^4 > 0.$$

If R is not nef, then from (1.2) it follows that

(3.8) There exists a rational solution (u_0, v_0) of the system

$$(\xi_{\mathscr{E}} + uD)^2(\xi_{\mathscr{E}} + vD)\xi_{\mathscr{E}} = 0,$$
$$(\xi_{\mathscr{E}} + uD)^2(\xi_{\mathscr{E}} + vD)D = 0.$$

We solve these equations using relations (3.1) and (3.2). We assume that c is positive and satisfies inequalities (3.3) through (3.5). It turns out that under these assumptions the problem (3.8) has no solution at all. On the other hand (3.7) has a solution only in the following three cases:

$$\text{(i)} \quad M = P^3, \quad c = 1, \quad u_0 = 1;$$
$$\text{(ii)} \quad M = Q^3, \quad c = 1, \quad u_0 = 1;$$
$$\text{(iii)} \quad M = V_4, \quad c = 1, \quad u_0 = \tfrac{1}{2}.$$

We rule out the last case.

(3.9) **Lemma.** *If $\mathscr{E}$ is a rank-2 vector bundle on V_4 such that $c_1\mathscr{E} = 0$ and $c_2\mathscr{E} = ($ class of the line on $V_4)$, then $\mathscr{E}$ is not Fano.*

Proof. From the above discussion it follows that if $\mathscr{E}$ is Fano then the extremal ray R of $P(\mathscr{E})$ (which is not contracted by p) is nef and $D_R = 2\xi_{\mathscr{E}} + D$ is a good supporting divisor for R. On the other hand, as in the proof of (3.6), we see that $H^0(V_4, \mathscr{E}) \neq 0$ and therefore $\xi_{\mathscr{E}}$ is an effective divisor on $P(\mathscr{E})$. Now if $D_R = 2\xi_{\mathscr{E}} + D$ is a good supporting divisor for the ray R then, since $D \cdot R > 0$, we get $\xi_{\mathscr{E}} \cdot R < 0$ hence R cannot be nef.

Similar argument as in the above proof shows that the bundles in cases (i) and (ii) are Fano only if they have no sections, that is, are stable. A stable rank-2 bundle on P^3 with $c_1 = 0$, $c_2 = 1$ is called the null-correlation bundle, or simply NCB. In [SzW2] we show that its projectivization is a Fano 4-fold which is also a projectivization of a rank-2 stable vector bundle on Q^3 with $c_1 = -D$, $c_2 = 1$.

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