INFINITESIMAL PSEUDO-METRICS AND
THE SCHWARZ LEMMA

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ABSTRACT. In this paper we investigate the relationship between infinitesimal pseudo-metrics introduced by N. Sibony and K. Azukawa. Also we prove a version of the Schwarz lemma for plurisubharmonic functions.

INTRODUCTION

Let $M$ be a complex manifold. Throughout the paper we shall be assuming that the dimension of $M$ is $n$. Sibony in [8] and Azukawa in [1] introduced infinitesimal pseudo-metrics on the tangent bundle $TM$ using families of bounded plurisubharmonic functions on $M$. We shall denote the pseudo-metrics by $S_M$ and $A_M$ respectively (the definitions of the pseudo-metrics are given in the next section). Both $S_M$ and $A_M$ contract holomorphic mappings and hence $C_M \leq S_M \leq K_M$ and $C_M \leq A_M \leq K_M$ where $C_M$ and $K_M$ denote the Carathéodory and Kobayashi infinitesimal pseudo-metrics respectively. The purpose of this paper is to study the relationship between $S_M$ and $A_M$.

In order to be able to state the main results we need to recall the definition of an extremal plurisubharmonic function, originally introduced in [6] and then studied in [2, 4].

Let $\mathcal{P}(M, p)$ denote the family of all negative plurisubharmonic functions $u$ on $M$ such that for any holomorphic chart $\varphi: U \to \varphi(U) \subset \mathbb{C}^n$ where $p \in U \subset M$ and $\varphi(p) = 0$, the function $u \circ \varphi^{-1} - \log \| \cdot \|$ is bounded from above in a neighbourhood of 0. (By $\| \cdot \|$ we denote the Euclidean norm in $\mathbb{C}^n$.) Define

$$u_M(z, p) = \sup\{u(z) : u \in \mathcal{P}(M, p)\}.$$ 

It can be proved that $u_M(\cdot, p) \in \mathcal{P}(M, p)$. Moreover, if $M$ is an open subset of $\mathbb{C}$ such that $u_M(z, p) \to 0$ as $z \to \partial M$, then $-u_M(\cdot, p)$ coincides with the Green function for $M$ with pole at $p$. For further properties of the extremal function $u_M$ and its applications in complex analysis see [6, 4, 2].
Let $\mathcal{C}^2(\{p\})$ denote the family of all functions which are of class $\mathcal{C}^2$ in some neighbourhood of $p$.

We show the following.

**Theorem 1.** If $M$ is a complex manifold then $S_M \leq A_M$. If $M$ is a Stein manifold then $A^*_M$ is upper semicontinuous and hence $S_M^* \leq A_M$ (where the asterisk denotes the upper-semicontinuous regularization). If $\exp(2u_M(\cdot, p)) \in \mathcal{C}^2(\{p\})$ for some $p \in M$ then

\[(2)\quad S_M(\xi) = A_M(\xi) = \langle \mathcal{L}(\exp(2u_M(\cdot, p)))(p)\xi, \xi \rangle^{1/2}, \quad \xi \in T_pM,
\]

where $T_pM$ is the tangent space to $M$ at $p$ and $\langle \mathcal{L}', \cdot \rangle$ denotes the Levi form.

If $P_M$ is an infinitesimal pseudometric on $M$ and $p \in M$, the set $I_p(P_M) = \{v \in T_pM : P_M(v) < 1\}$ is called the indicatrix of $P_M$ at $p$. It can be shown that the indicatrices of $S_M$ are always convex (see [8]) and those of $A_M$ are—in general—only starlike circular. Therefore, it is easy to furnish examples of manifolds $M$ for which the two pseudo-metrics differ. For instance we can take a plurisubharmonic function $h: \mathbb{C}^n \to \mathbb{R}^+$ such that $h(\lambda z) = |\lambda|h(z)$ for $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^n$ and $M = \{z \in \mathbb{C}^n : h(z) < 1\}$ is not convex (e.g. $h(z_1, z_2) = \max\{|z_1|, |z_2|, 2\sqrt{|z_1z_2|}\}$ for $(z_1, z_2) \in \mathbb{C}^2$). Then $A_M(0, \xi) = h(\xi)$ (see [1]) and hence $A_M \neq S_M$ on $T_0M \equiv \mathbb{C}^n$. An example of a nonconvex $M$ for which $S_M = A_M$ is provided by the open annulus $M = \{z \in \mathbb{C} : r < |z| < R\}$ where $r > 0$, $R > 0$. The equality follows from Theorem 1 because $\exp(2u_M(\cdot, p)) \in \mathcal{C}^2(M)$ (see [6]).

We also prove a version of the Schwarz lemma for plurisubharmonic functions. To state the result, we have to introduce some notation. Let $p \in M$ and let $S^p(M, p)$ denote the family of all logarithmically plurisubharmonic functions $u: M \to [0, 1]$ such that $u(p) = 0$ and $u \in \mathcal{C}^2(\{p\})$. By $\mathcal{F}(M, p)$ we will denote the set of all plurisubharmonic functions $u$ on $M$ with the property that for each $z \in M\{p\}$ there exists a connected one-dimensional complex submanifold $N$ of $M$ such that $z, p \in N$ and the restriction of $u$ to $N\{p\}$ is harmonic.

**Theorem 2.** Let $\Omega$ be a relatively compact open connected subset of a Stein manifold $M$ and let $u_\Omega = u(\cdot, p)$. If $v \in S^p(\Omega, p)$ then

\[(3)\quad v \leq \exp(2u_\Omega) \quad \text{in} \Omega\{p\}.
\]

Moreover if $\exp(2u_\Omega) \in \mathcal{C}^2(\{p\})$ then for all $\xi \in T_pM$

\[(4)\quad \langle \mathcal{L}v(p)\xi, \xi \rangle \leq \langle \mathcal{L}(\exp(2u_\Omega))(p)\xi, \xi \rangle.
\]

If $u_\Omega \in \mathcal{F}(\Omega, p)$ and the equality holds in (4) for all $\xi \in T_pM$ then $v \equiv \exp(2u_\Omega)$. If $n = 1$ and the equality holds in (3) for one $z \in \Omega$ then $v \equiv \exp(2u_\Omega)$.

When $\Omega$ is equal to the unit disc, $\exp(2u_\Omega(z)) = |z|^2$ and hence the theorem reduces to the version of the Schwarz lemma proved by Sibony in [8].
It is easy to notice that the last conclusion of the theorem is not true for
\( n > 1 \). For instance, if \( \Omega \) is the open unit ball in \( \mathbb{C}^2 \), \( \mathbf{p} \) is the origin and
\( v(z_1, z_2) = |z_1|^2 + a|z_2|^2 \) (where \( a \in [0, 1) \) is fixed) then \( v \neq \exp(2u_\Omega) = \| \cdot \|^2 \).

Note also that if \( n = 1 \), \( \exp 2u_\Omega \in \mathcal{S}^2(\{ \mathbf{p} \}) \). This is so, because \( -u_\Omega \) is the
generalized Green function for \( \Omega \) with pole at \( \mathbf{p} \).

Assume now that \( \Omega \subset M \) is such that \( u = u_\Omega(\cdot, \mathbf{p}) \) is a \( \mathcal{S}^2 \)-function on
\( \Omega \setminus \{ \mathbf{p} \} \) and \( u(z) \to 0 \) as \( z \) approaches the boundary of \( \Omega \). It is known that \( u \)
satisfies the homogeneous Monge-Ampère equation \( (dd^c u)^n = 0 \) in \( \Omega \setminus \{ \mathbf{p} \} \) (see
[6]). If we also assume that \( (dd^c u)^{n-1} \neq 0 \) at each point of \( \Omega \setminus \{ \mathbf{p} \} \), then—in
view of [3]—there is a foliation of \( \Omega \setminus \{ \mathbf{p} \} \) by one-dimensional complex mani-
folds with the property that the restriction of \( u \) to each of them is harmonic (an
alternative proof can be found in [5]). As observed in [4], it follows from the
maximum principle that if \( N \) is a leaf of the foliation, then \( \mathbf{p} \in \overline{N} \) and \( \overline{N} \cap \Omega \)
is a one-dimensional analytic subvariety of \( \Omega \). It would be interesting to know
under what condition \( \mathbf{p} \) is a regular point of \( \overline{N} \cap \Omega \) for each leaf \( N \) of the
foliation. (If this was the case, \( u \) would be a member of \( \mathcal{S}(\Omega, \mathbf{p}) \).) Under
the assumption that \( \Omega \) is a bounded strictly convex domain in \( \mathbb{C}^n \), an affirmative
answer follows from Lempert’s study of the Kobayashi metric [7]. It has been
conjectured by Demailly [4] that if \( \Omega \) is strictly pseudoconvex then the above
conditions are also met—i.e. \( u \) is smooth on \( \Omega \setminus \{ \mathbf{p} \} \), \( dd^c u \) has constant rank
\( n - 1 \) in \( \Omega \setminus \{ \mathbf{p} \} \) and the leaves of the associated foliation extend through \( \mathbf{p} \) to
complex submanifolds of \( \Omega \).

In the light of the above remarks, the assumption that \( u_\Omega \in \mathcal{S}(\Omega, \mathbf{p}) \) does
not seem to be too restrictive.

1. The families of functions \( \mathcal{P}(M, \mathbf{p}) \) and \( \mathcal{S}(M, \mathbf{p}) \)

In this section we shall establish the relationship between the two families of
plurisubharmonic functions defined in the introduction.

The following lemma follows directly from the proof of a version of the
Schwarz lemma obtained by Sibony in [8]. For the sake of completeness we
give a proof here.

**Lemma 1.** Let \( D \subset \mathbb{C} \) be a neighbourhood of zero. If \( u \in \mathcal{P}(D, 0) \) then
\begin{equation}
(5) \quad u(z) = \frac{1}{4}(\Delta u)(0)|z|^2 + o(|z|^2), \quad \text{as } z \to 0.
\end{equation}

**Proof.** In view of Taylor’s formula for \( u \) at \( 0, v(z) = (u(z)/|z|^2)^* \) is subharmonic in \( D \) and
\[
\lim_{t \to 0^+} \frac{u(t\alpha, t\beta)}{t^2} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(0)\alpha^2 + \frac{\partial^2 u}{\partial x \partial y}(0)\alpha\beta + \frac{1}{2} \frac{\partial^2 u}{\partial y^2}(0)\beta^2
\]
for any \( \alpha + i\beta \) from the unit circle. Moreover, as closed line segments in \( \mathbb{C} \)
are not thin, the limit on the left hand side of the above equality is \( v(0) \). By
taking $\alpha + i\beta$ equal to 1, $i$ and $(1 + i)/\sqrt{2}$ we conclude that

$$\frac{\partial^2 u}{\partial x^2}(0) = \frac{\partial^2 u}{\partial y^2}(0) \quad \text{and} \quad \frac{\partial^2 u}{\partial x\partial y}(0) = 0.$$

Hence the Taylor expansion of $u$ at 0 looks exactly as stated in our lemma.

Let $M$ be a complex manifold and let $p \in M$. For $u \in \mathcal{C}^2(\{p\})$ one can define the Levi form $\langle \mathcal{L}u(p) , \cdot \rangle$ of $u$ at $p$ as follows. Let $\xi \in T_pM$ and let $\varphi : U \to \varphi(U) \subset \mathbb{C}^n$ be a chart on $M$ in a neighbourhood $U$ of $p$ such that $\varphi(p) = 0$. If $(\zeta_1, \ldots, \zeta_n) = d_p\varphi(\xi)$ then we put

$$\langle \mathcal{L}u(p)\xi , \xi \rangle = \sum_{i,j=1}^{n} \frac{\partial^2(u \circ \varphi^{-1})(0)}{\partial z_i\partial \bar{z}_j} \zeta_i\bar{\zeta}_j.$$

It is easy to show that this definition is independent of the choice of $\varphi$.

**Lemma 2.** If $u \in \mathcal{S}_p(M, p)$ then $\log \sqrt{u} \in \mathcal{P}(M, p)$. Moreover if $V \subset \mathbb{C}$ is a neighbourhood of the origin and $F : V \to M$ is a holomorphic mapping such that $F(0) = p$ and $F'(0) = \xi$ we have

$$\langle \mathcal{L}u(p)\xi , \xi \rangle = \lim_{\lambda \to 0} \frac{(u \circ F)(\lambda)}{|\lambda|^2}.$$

**Proof.** Without loss of generality we may assume that $M$ is an open subset of $\mathbb{C}^n$ and $p = 0$. Since $\langle \mathcal{L}u(p)\xi , \xi \rangle = \frac{1}{2}\Delta(\lambda \to (u \circ F)(\lambda)|_{\lambda=0}$ (where $\lambda \in \mathbb{C}$), (5) implies that in a neighbourhood of the origin in $\mathbb{C}$

$$\langle u \circ F(\lambda) = (\mathcal{L}u(0)\xi , \xi)|\lambda|^2 + o(|\lambda|^2).$$

(6) is implied directly by (7). Also from (7), applied to $F(\lambda) = \lambda\xi$, we deduce that Taylor's expansion of $u$ about the origin in $\mathbb{C}^n$ has the following form:

$$u(z) = \langle \mathcal{L}u(0)z , z \rangle + o(\|z\|^2).$$

The first conclusion of the lemma follows form (8).

It is interesting to notice that if $u \in \mathcal{PSH}(\mathbb{C}^n) \cap \mathcal{C}^2(\{0\})$ is such that

$$u(\lambda z) = |\lambda|^2 u(z)$$

for all $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^n$, then $u(z) = \langle \mathcal{L}u(0)z , z \rangle$ for all $z \in \mathbb{C}^n$ and hence $u^{1/2}$ is a seminorm. To see this, it is enough to apply the Laplace operator (with respect to $\lambda$) to both sides of (†). (See also the remarks following Theorem 1 in the Introduction.)

The families $\mathcal{S}_p(M, p)$ and $\mathcal{P}(M, p)$ have been used in the definitions of $S_M$ and $A_M$ respectively (see [8, 1, 2]). $S_M$ is defined by the formula

$$S_M(\xi) = \sup\{\langle \mathcal{L}u(p)\xi , \xi \rangle^{1/2} : u \in \mathcal{S}_p(M, p)\}, \quad \xi \in T_pM.$$

For more information about the pseudo-metric $S_M$ see [8, 9].
Assume \( \xi \in T_pM \). Let \( V \subset \mathbb{C} \) be a neighbourhood of the origin and let \( F : V \to M \) be a holomorphic mapping such that \( F(0) = p \) and \( F'(0) = \xi \). Then we put
\[
L_u[\xi] = \limsup_{\lambda \to 0} \frac{\exp(u \circ F)(\lambda)}{\lambda}
\]
for any \( u \in \mathcal{P}(M, p) \). It can be proved that the above definition is independent of the choice of \( F \) (see [1, 2]). (If \( \exp(2u) \in \mathcal{S}^2(\{p\}) \), it follows directly from (6).)

Following Azukawa [1, 2] we define
\[
A_M(\xi) = \sup_{u \in \mathcal{P}(M, p)} \{ L_u[\xi] \}, \quad \xi \in T_pM.
\]
From the definition of the extremal function \( u_M = u_M(\cdot, p) \) we conclude (as in [2]) that for \( \xi \in T_pM \)
\[
A_M(\xi) = L_{u_M}[\xi].
\]

2. Semicontinuity of \( A_M \)

In this section we shall prove that if \( M \) is a Stein manifold then \( A_M \) is upper semicontinuous.

**Lemma 3.** If \( M \) is a Stein manifold then \( u_M : M \times M \to [-\infty, 0) \) is upper semicontinuous.

**Proof.** Since \( M \) is a Stein manifold, there exists a smooth plurisubharmonic function \( \psi : M \to \mathbb{R} \) such that the set \( M_c^{\text{def}} = \{ z \in M : \psi(z) < c \} \) is relatively compact in \( M \) for each \( c \in \mathbb{R} \). It follows from the definition of the extremal function \( u_M \) that the sequence \( \{ u_M \}_{j \in \mathbb{N}} \) is decreasing and \( \lim_{j \to \infty} u_{M_j} = u_M \) (see also [2] and [4]). Furthermore, as each of the sets \( M_j \) is hyperconvex (in the sense of [4]), the functions \( u_{M_j} : M_j \times M_j \to [-\infty, 0) \) are continuous by Theorem 4.14 in [4]. This means that the limit function \( u_M \) is upper semicontinuous.

**Lemma 4.** If \( M \) is a complex manifold such that \( u_M : M \times M \to [-\infty, 0) \) is upper semicontinuous then \( A_M : TM \to [0, \infty) \) is also upper semicontinuous.

**Proof.** Take \( \xi_0 \in TM \). Let \( \pi : TM \to M \) be the canonical projection (i.e. \( \pi(\xi) = q \Leftrightarrow \xi \in T_qM \)). Let \( \varphi : U \to \varphi(U) \subset \mathbb{C}^n \) be a holomorphic chart in a neighbourhood \( U \) of \( \pi(\xi_0) = p \). The chart \( \varphi \) generates a chart \( \hat{\varphi} \) on \( TM \) in the following way:

\[
\hat{\varphi} : \pi^{-1}(U) \to \varphi(U) \times \mathbb{C}^n,
\]
\[
\hat{\varphi}(\xi) = ((\varphi \circ \pi)(\xi), a_1, \ldots, a_n) \leftrightarrow \xi = \sum_{j=1}^n a_j \frac{\partial}{\partial \varphi_j} \mid_{\pi(\xi)}.
\]
Define $g : \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ by the formula $g(\lambda, z, a) = \lambda a + z$ and set $F(\lambda, \xi) = \varphi^{-1}(g(\lambda, \varphi(\xi)))$. Take a neighbourhood $V_0$ of $\xi_0$ and $r_0 > 0$ such that the mapping $\lambda \to F(\lambda, \xi)$ is well defined if $|\lambda| < r_0$ and $\xi$ is fixed in $V_0$. Let $F_\xi$ denote this mapping. Clearly $F_\xi(0) = \pi(\xi)$. In particular $F_{\xi_0}(0) = \pi$. Notice that if

$$\xi = \sum_{j=1}^n a_j \frac{\partial}{\partial \varphi_j} |_{\pi(\xi)},$$

then $(d_{\pi(\xi)} \varphi)(\xi) = (a_1, \ldots, a_n)$. Therefore

$$d_0 F_\xi = (d_{\pi(\xi)} \varphi)^{-1} \circ d_0(\lambda \to g(\lambda, \varphi(\xi)))$$

and hence

$$F_\xi'(0) = d_0 F_\xi \left( \frac{d}{d\lambda} |_{0} \right) = (d_{\pi(\xi)} \varphi)^{-1}(a_1, \ldots, a_n) = \xi.$$

Thus

$$(12) \quad A_M(\xi) = \limsup_{\lambda \to 0 \atop \lambda \in \mathbb{C}} \frac{\exp u_M(F_\xi(\lambda), \pi(\xi))}{|\lambda|}, \quad \xi \in \pi^{-1}(U).$$

Suppose $A_M(\xi_0) < c$. By (12), there is a number $r \in (0, r_0)$ such that

$$\sup_{|\lambda|=r} u_M(F_\xi(\lambda), \pi(\xi_0)) - \log |\lambda| < \log c.$$

This means that

$$\sup_{|\lambda|=r} u_M(F_\xi(\lambda), \pi(\xi_0)) < \log rc.$$

Because of upper semicontinuity of $u_M$ and continuity of $F$ and $\pi$ one can find a neighbourhood $V$ of $\xi_0$ such that $V \subset V_0$ and

$$\sup_{|\lambda|=r} u_M(F_\xi(\lambda), \pi(\xi)) < \log rc, \quad \xi \in V.$$

Therefore

$$\sup_{|\lambda|=r} (u_M(F_\xi(\lambda), \pi(\xi)) - \log |\lambda|) < \log c.$$

If $\xi$ fixed in $V$, the function $(\lambda \to u_M(F_\xi(\lambda), \pi(\xi)) - \log |\lambda|)^*$ is subharmonic in the disc $\{\lambda : |\lambda| < r_0\}$. Thus by the maximum principle for subharmonic functions

$$\limsup_{\lambda \to 0 \atop \lambda \neq 0} (u_M(F_\xi(\lambda), \pi(\xi)) - \log |\lambda|) < \log c$$

for every $\xi \in V$. Consequently $A_M < c$ in $V$.

3. Proof of the theorems

The first statement of Theorem 1 follows directly from the definitions (9), (11) of the pseudo-metrics and from Lemma 2. Lemma 3 and Lemma 4 yield the second conclusion of the theorem.
The estimates (3), (4) in Theorem 2 follow from the definition of $u_\Omega$ and from Lemma 2.

Now assume that $u_\Omega \in \mathcal{F}(\Omega, p)$ and the equality holds in (4) for all $\zeta \in T_pM$. Without loss of generality we may assume that $M$ is submanifold of $\mathbb{C}^m$ for some $m$. Since $\Omega$ is relatively compact in $M$, it is contained in an open ball with centre at $p$ and a positive radius $r$. It is clear that $\log ||z - p|| - \log r \leq u_\Omega(z)$ for all $z \in \Omega$. Let $N$ be a connected one-dimensional complex submanifold of $\Omega$ such that $p \in N$ and $u|N\{p\}$ is harmonic. Let $F : V \to M$ be a holomorphic parametrization of $N$ in a neighbourhood of $p$, such that $0 \in V \subset \mathbb{C}$ and $F(0) = p$. By applying (6) and the lower estimate for $u_\Omega$, we get

$$<\mathcal{L}(\exp 2u_\Omega)(p) F'(0), F'(0)> \geq \lim_{\lambda \to 0} \frac{\|F(\lambda) - F(0)\|^2}{|\lambda|^2} = \frac{\|F'(0)\|^2}{r^2} > 0.$$

Hence, (6) implies that

$$1 = \frac{<\mathcal{L} v(p) F'(0), F'(0)>}{<\mathcal{L}(\exp 2u_\Omega)(p) F'(0), F'(0)>>} = \lim_{\lambda \to 0} \frac{(v \circ F)(\lambda)}{\exp(2u_\Omega \circ F)(\lambda)}.$$

Therefore the function $((v/\exp 2u_\Omega)|N)^*$ is subharmonic on $N$ and attains its maximal value at $p$. Thus the maximum principle implies that $v = \exp(2u_\Omega)$ on $N$. As $u_\Omega \in \mathcal{F}(\Omega, p)$, the same equality holds in $\Omega$.

If $n = 1$ and the equality holds in (3) for one point $z \in \Omega\{p\}$, then the subharmonic function $v/(\exp 2u_\Omega)$ has its maximum at $z$. Therefore—the by the maximum principle—it is constant.

**References**